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ANALYSIS

Theory of Functions of Complex Variables

Kennedy, E. S. Exponential analogues of the Lambert series. Amer. J. Math. 63, 443–460 (1941). [MF 4164]

Let z denote a complex variable, a_n, λ_n constants, λ_n being real, $\lambda_n<\lambda_{n+1}$, $\lambda_n\to\infty$. The author calls "R-series" a series of the type

$$\sum_{n=1}^{\infty}\frac{a_n e^{-\lambda_n z}}{1-e^{-\lambda_n z}}.$$

If $\lambda_n=\log n$, and the term $n=1$ is omitted, the series is called an "ordinary" R-series. The case $\lambda_n=n$, that of the Lambert series (it is more usual to write it with x for e^{-z}), has often been considered. The paper is concerned with analogues and extensions of well-known results on Lambert series to "general" or to "ordinary" R-series.

G. Pólya (Providence, R. I.).

v. Sz. Nagy, Gyula. Über ganze Funktionen mit lauter reellen Nullstellen. Univ. Nac. Tucumán. Revista A. 1, 303–311 (1940). [MF 4070]

The author obtains a generalization of a theorem of E. Laguerre on polynomials with real zeros. Let $f(z)$ be an entire function of genus zero having only real zeros. Let α and β ($\beta>\alpha$) be two consecutive zeros of multiplicity q and r, respectively. Let $\alpha_1, \alpha_2, \cdots$ be the zeros of $f(z)$ less than α and $\beta_1, \beta_2, \cdots$ those greater than β (each zero being counted as often as its multiplicity). Then the zero z_0 of the derivative $f'(z)$ which lies on the segment (α, β) of the real axis satisfies the inequalities

$$\alpha+(\beta-\alpha)q/(q+S_\beta)<z_0<\beta-(\beta-\alpha)r/(r+S_\alpha),$$

where

$$S_\alpha=q+\textstyle\sum_k(\beta-\alpha)(\beta-\alpha_k)^{-1},$$
$$S_\beta=r+\textstyle\sum_k(\beta-\alpha)(\beta_k-\alpha)^{-1}.$$

Next suppose $f(z)$ is an entire function of genus zero which is real on the real axis. Let α and β ($\beta>\alpha$) be real zeros of $f(z)$ of multiplicity q and r, respectively. Suppose $f(z)$ has no zeros inside the circle with diameter (α, β). Let $a_1, a_2, \cdots$ be the zeros of $f(z)$ whose real parts are less than β and $\beta_1, \beta_2, \cdots$ those zeros whose real parts are greater than α. Let c_k be the diameter of the circle through the points β, α_k, $\bar{\alpha}_k$, where $\bar{\alpha}_k$ denotes the conjugate of α_k. Similarly let d_k be the diameter of the circle through the points α, β_k, $\bar{\beta}_k$. Then no zero of the derivative $f'(z)$ lies inside the intervals

$$(\alpha, \alpha+(\beta-\alpha)q(q+S_\beta)^{-1}),\quad (\beta-(\beta-\alpha)r(r+S_\alpha)^{-1}, \beta),$$

where

$$S_\alpha=q+\textstyle\sum_k(\beta-\alpha)c_k^{-1},\quad S_\beta=r+\sum_k(\beta-\alpha)d_k^{-1}.$$

The author obtains further analogous theorems for simple entire functions of genus p which have only real zeros and which do not vanish at the origin.

M. S. Robertson (New Brunswick, N. J.).

Weisner, Louis. Power series the roots of whose partial sums lie in a sector. Bull. Amer. Math. Soc. 47, 160–163 (1941). [MF 3834]

Pólya [1913] proved: If the roots of the partial sums of a convergent power-series $f(z)=\sum a_n z^n$ lie in a sector with vertex at the origin and aperture $\alpha<\pi$, the order of the integral transcendental function $f(z)$ is not greater than 1. Pólya [1913] also proved that, in case $\alpha=0$, the order of the function is 0. Weisner combines and extends these results in the following theorem: If, for each $n\geqq n_0$, the roots of the partial sum of degree n of the formal power-series $f(z)=\sum_0^\infty a_n z^n$ lie in some sector with vertex at the origin and aperture $\alpha<\pi$, then $f(z)$ is an integral transcendental function of order 0. It is stated that, while α is independent of n, the lines bounding the sector may be different for different values of n. The proof is furnished by skillful use of elementary inequalities.

A. J. Kempner (Boulder, Colo.).

Kamenetzky, I. M. Sur l'indicatrice de croissance d'une fonction entière du premier ordre et sur la distribution des singularités d'une fonction représentée par la série de Taylor associée. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 321–323 (1940). [MF 3246]

Introducing the function

$$A(\lambda, \alpha)=\max[A_1(\lambda, \alpha), \cdots, A_m(\lambda, \alpha)],$$

where

$$A_i(\lambda, \alpha)=(\lambda^2+2\lambda\lambda_i\cos(\alpha-\alpha_i)+\lambda_i^2)^{\frac{1}{2}},\quad i=1, 2, \cdots, m,$$

the author proves that a necessary and sufficient condition for the function

$$h(\phi)=\max[\lambda, \cos(\phi+\alpha_1), \cdots, \lambda_m\cos(\phi+\alpha_m)]$$

to be the indicatrix of growth for the entire function $f(z)=\sum a_n z^n/n!$ of the first order and of the normal type is that the equality

$$\overline{\lim_{n=\infty}}\,(|\textstyle\sum C_n^k t^{n-k}a_k|)^{1/n}=A(\lambda, \alpha)$$

holds for every $t=\lambda e^{i\alpha}$ ($\lambda\geqq 0$). It is, therefore, also a necessary and sufficient condition for the smallest convex polygonal line containing the points $t_j=\lambda_j e^{i\alpha_j}$ to be the convex region of singularities for $F(z)=\sum a_n/z^{n+1}$. [Note of reviewer: these results should be compared with those of Denjoy, Ann. École Norm. 55, 257–336 (1939).]

S. Mandelbrojt (Houston, Tex.).

Ketchum, P. W. On the possible rate of growth of an analytic function. Trans. Amer. Math. Soc. 49, 211–228 (1941). [MF 3922]

A known expansion theorem states that for any set of n points $a_1, a_2, \cdots, a_n$ there exists a sequence of functions $\phi_k(z)$, $k=1, 2, \cdots$, such that $\phi_k(z)$ is analytic in a region including all the a's and such that, for any function $f(z)$

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analytic at each of the a's,

$$f(z)=\sum_{k=1}^{\infty} c_k\phi_k(z),$$

where the c's are constant. By making suitable changes the author extends this result to the case where n is infinite.

This generalization requires the proof of the following results: Given a sequence of points in the complex plane, $a_1, a_2, \cdots$, with the point at infinity as the only limit point. Then it is possible to find regions $S_1, S_2, \cdots$ with a_i interior to S_i such that, no matter how fast the numbers m_i increase, there is an entire function $f(z)$ such that $|f(z)|\geqq m_i$ in S_i. This result is a special case of a more general theorem proved by the author. The author also considers topological questions in connection with the regions S_i. *N. Levinson* (Cambridge, Mass.).

Levinson, Norman. A theorem of Boas. Duke Math. J. **8**, 181–182 (1941). [MF 3953]

The author gives a simple proof of a recent theorem of R. P. Boas [Duke Math. J. **6**, 719–721 (1940); these Rev. **2**, 82]. It may be observed that the method of proof is the same as that used by M. Ålander [Sur le déplacement des zéros des fonctions entières par leur dérivation, Thesis, Upsala, 1914; see pp. 6–8] and that it may be easily extended to yield the following more general result: An entire function $f(z)$ of order λ possesses an infinity of derivatives $f^{(n)}(z)$ which are univalent in a circle of given center (e.g., $z=0$) whose radius is of the order $n^{(1-\lambda)/\lambda}$.

G. Pólya (Providence, R. I.).

de Bruijn, N. G. Ein Satz über schlichte Funktionen. Nederl. Akad. Wetensch., Proc. **44**, 47–49 (1941). [MF 4044]

Let

$$w=f(z)=a_1z+a_2z^2+a_3z^3+\cdots$$

be regular and univalent in $|z|<1$. Let the image G in the w-plane of the unit circle have the property that each straight line through a fixed point a have at most only one segment of this line in common with G. If the point a is inside G, then G will be star-like with respect to a. R. Nevanlinna [Acta Soc. Sci. Fennicae. Nova Ser. A. **63**, no. 6 (1920–1921)] showed that $|a_n|\leqq n|a_1|$ $(n=1, 2, \cdots)$ in the particular case when a is the origin in the w-plane. M. S. Robertson [Amer. J. Math. **58**, 465–472 (1936)] obtained the same inequalities $|a_n|\leqq n|a_1|$ when a is the point at infinity, in which case G may be said to be convex in one direction. The author generalizes these results to arbitrary points a. He obtains for a real number θ

$$|a_{n+2}e^{(n+2)i\theta}-a_ne^{ni\theta}|\leqq 2|a_1|, \quad n=0, 1, 2, \cdots,$$

$$|a_n|\leqq n|a_1|.$$

M. S. Robertson (New Brunswick, N. J.).

Macintyre, A. J. and Wilson, R. Coefficient density and the distribution of singular points on the circle of convergence. Proc. London Math. Soc. (2) **47**, 60–80 (1940). [MF 3651]

A close study of the connection between the distribution of the vanishing coefficients of a power series and the distribution of the singular points of the function represented by the power series leads to the general conclusion that the influence of different kinds of singularities on the vanishing of coefficients can be suitably expressed by the introduction of different kinds of densities. This is confirmed from a new and interesting point of view by the present paper. One of the results of the paper can be stated (in a slightly weakened form) as follows: If a power series has only two singular points on its circle of convergence, both of them isolated and not critical, the density of the vanishing coefficients is zero, unless the singularities are placed at two of the vertices of a regular polygon inscribed in the circle of convergence. Now, if the singularities are placed in this way and if among all corresponding regular polygons that with the smallest number of sides is q-sided, then the upper density of the vanishing coefficients is at most $1/q$. It follows from another result of the authors that the theorem just stated remains true if the terms "isolated and not critical," "density" and "upper density" are replaced by "of easily approachable type," "lower density" and "lower density," respectively. The authors have analogous but less complete results for more than two singular points on the circle of convergence. *G. Pólya* (Providence, R. I.).

Montel, Paul. Sur la distance des points en lesquels une fonction analytique prend des valeurs données. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 31–39 (1940). [MF 4401]

A function $f(z)$, holomorphic in the unit circle and having at the center the value a, can take a different value b only at points whose distance from the center of the circle exceeds a positive number which depends only upon a, b and the maximum M of the modulus of $f(z)$ in the circle. The author points out that this result of E. Landau holds for every normal family. If a family of functions holomorphic in the unit circle is normal in the interior of this circle, for each domain completely interior to the circle there exists a positive number δ such that the distance of two points at which the function takes on different fixed values a and b is greater or equal to δ.

The author obtains the exact value of this minimum distance for a fairly general class of functions $f(z)=a+a_1z+a_2z^2+\cdots$ for which

$$\sum_{n=1}^{\infty}|a_nb_n|^{\alpha}\leqq R^{\alpha},$$

R being a fixed number, $\alpha>1$, and b_n a sequence of positive numbers for which $\liminf (b_n)^{1/n}\geqq 1$. [The author has the limit superior which seems to be incorrect.] If $b\neq a$ is a value of $f(z)$, the minimum distance δ for the class is the positive root of the equation

$$\phi(\delta^{\beta})=\left|\frac{b-a}{R}\right|^{\beta}, \quad \frac{1}{\alpha}+\frac{1}{\beta}=1,$$

where

$$\phi(z)=\sum_{n=1}^{\infty}b_n^{-\beta}z^n.$$

The distance δ is attained for the function

$$f_0(z)=a+\frac{R^{\beta}}{|b-a|^{\beta-1}}\phi(\delta^{\beta-1}z).$$

The author calculates δ explicitly for several special cases. The result is extended to the case $\alpha=1$ when each $b_n=1$, and also to the case where α increases indefinitely.

The family of derivative functions $f'(z)$ for the class under consideration is normal and bounded in the interior of the unit circle. The author obtains the exact limit of the

maximum of the modulus of $f'(z)$ for $|z| \leqq \rho < 1$. If

$$\sum_{n=1}^{\infty} |a_n b_n|^{\alpha} = R^{\alpha},$$

R finite but varying with $f(z)$ and not bounded for the class, then the family is quasi-normal if, for a positive fixed λ, $|f(0)| \geqq \lambda R$. The author finds that the family is normal in a concentric circle and he shows how the exact value of its radius may be obtained.

M. S. Robertson (New Brunswick, N. J.).

Heins, Maurice H. On the iteration of functions which are analytic and single-valued in a given multiply-connected region. Amer. J. Math. **63**, 461–480 (1941). [MF 4165]

Let G_z be an arbitrarily multiply-connected region of the z-plane, the boundary of which contains at least three points. The function $w=f(z)$ is analytic and single-valued for $z \subset G_z$, and $z \subset G_z$ implies $w \subset G_z$. If $f_0(z)$ is defined as $f(z)$ and $f_n(z)$ as $f(f_{n-1}(z))$, the asymptotic behavior of the sequence of iterates $\{f_n(z)\}$ as $n \to \infty$ is the problem considered by the author in this paper. The same problem in the case G_z is the unit circle has been solved by Denjoy and Wolff; the present paper generalizes their results and obtains some results of a different character due to the introduction of multiply-connected regions.

If $w=f(z)$ is analytic near z_0, and if $f(z_0)=z_0$, then z_0 is said to be a fixed point of the map $w=f(z)$. If $|f'(z_0)|<1$, z_0 is said to be an attractive fixed point. The principle results of the paper are concisely summed up in the author's theorem 2.2: Under the assumptions on $f(z)$ and G_z made above, one and only one of the following mutually exclusive possibilities can occur. (1) $w=f(z)$ is a conformal map of G_z onto itself. (2) $w=f(z)$ has an attractive fixed point $\zeta \subset G_z$, and $\{f_n(z)\}$ converges continuously to ζ for all z in G_z. (3) The sequence $\{f_n(z)\}$ converges continuously to the boundary of G_z; that is, every convergent subsequence of $\{f_n(z)\}$ converges to a boundary point of G_z. The set of boundary points thus approached is either a single point or a continuum. In the case the boundary of G_z consists of $p < \infty$ Jordan curves, and if in addition case (3) of theorem 2.2 holds, the convergence of $\{f_n(z)\}$ is actually to a single point ζ of the boundary. The same is true if the boundary of G_z is totally discontinuous. In connection with this latter case, the author is able to clear up a conjecture made by Fatou in his prize-winning paper on the iteration of rational functions. Also in the case G_z is a region bounded by $p < \infty$ Jordan curves, the author is able to show that the limiting value ζ of the iterates $f_n(z)$ depends continuously on the function $f(z)$.

J. W. Green (Rochester, N. Y.).

Grunsky, Helmut. Eindeutige beschränkte Funktionen in mehrfach zusammenhängenden Gebieten. I. Jber. Deutsch. Math. Verein. **50**, 230–255 (1940).

Let $\mathfrak{B}$ denote a region of the z-plane of finite connectivity bounded by n disjoint continua. The author considers the extremal problems associated with analogues of the lemmas of Schwarz, Julia and Loewner for the region $\mathfrak{B}$. It is shown in the cases of the analogues of the lemmas of Julia and Loewner that the extremal functions are unique except for a pure imaginary additive constant, and that they define $(1, n)$ conformal maps of $\mathfrak{B}$ onto the corresponding image domains. A similar theorem is conjectured for the analogue of Schwarz's lemma, but is not established for $n>1$. The proofs are based upon a systematic use of the argument principle and of the fact that the set of $(1, m)$ maps (for all admissible m) of $\mathfrak{B}$ onto the image domain which satisfy the requirements of a generic function of the family of functions considered is dense in this family.

M. H. Heins (Princeton, N. J.).

af Hällström, Gunnar. Über meromorphe Funktionen mit mehrfach zusammengängenden Existenzgebieten. Acta Acad. Aboensis **12**, no. 8, 100 pp. (1940). [MF 3996]

The object of this paper is to show that the two fundamental theorems of R. Nevanlinna in the theory of functions meromorphic in a simply-connected region admit extension in modified form to the theory of functions which are single-valued and meromorphic in certain types of multiply-connected plane regions. The regions considered consist of the Riemann sphere deleted in a finite number of points or a denumerable set with at most one limit point. These extensions are made possible by a theorem of H. Selberg [Avh. Norske Vid. Akad. Oslo. I. **1937**, no. 10, 10 pp.] which affirms for plane regions G with boundaries of capacity zero the existence of a function $g(z)$ with the following properties: (1) $g(z)$ is single-valued and harmonic in G deleted at z_0, where z_0 is a point of G; (2) in the neighborhood of z_0, $g(z)$ differs from $\log|z-z_0|$ by a function which is harmonic at z_0; (3) as z tends to the boundary of G, $g(z) \to +\infty$ uniformly. The level curves of $g(z)$ replace the concentric circles of the classical theorems of Nevanlinna.

Following the procedure used by Ahlfors [Soc. Sci. Fenn. Comm. Phys.-Math. 8, no. 10, 14 pp. (1935)] to establish the theorems of Nevanlinna, the author establishes analogues of these theorems for the classes of multiply-connected regions which admit either a Green's function $\gamma(z)$ with the property that, whenever z tends to the boundary of the region, $\gamma(z)$ tends to zero, or else (in the case of a boundary of zero capacity) a function $g(z)$ of the type given by Selberg. These theorems are studied in greater detail for the special classes of multiply-connected regions with either a finite number of boundary points or else a denumerable set of boundary points with at most one limit point. The author concludes that the characteristic quantities involved in the analogues of the Nevanlinna theorems depend to a large degree upon the nature of $g(z)$. The "defect" relation is shown to remain valid in the case where the boundary of the region consists of a finite number of points. In the case where the boundary consists of a denumerably infinite set with a single limit point, only sufficient conditions that guarantee the "defect" relation are given. The author concludes by indicating several open questions.

M. H. Heins (Princeton, N. J.).

Sona, Luigi. Rappresentazione conforme di un piano con n tagli radiali su un piano forato circolarmente. Pont. Acad. Sci. Comment. **3**, 39–64 (1939). [MF 4100]

A method is given for calculating the constants involved in the mapping functions which map conformally and $(1, 1)$ the extended z-plane cut along n distinct rays of finite lengths issuing from the origin onto the exterior of the unit circle in the extended ζ-plane. Special symmetric cases are examined.

M. H. Heins (Princeton, N. J.).

Bieberbach, Ludwig. Schlitzabbildungen durch rationale Funktionen. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 143–148 (1940). [MF 4412]

The author gives a simple proof of the following theorem: The only rational functions $w=f(z)$ which define a $(1, 1)$

conformal map of $|z|<1$ onto a "Schlitzbereich" are of the form $w=U[T_s/(1-T_s)^2]$, where U and T are linear fractional transformations, the latter mapping $|z|<1$ onto itself. The theorem is generalized for certain classes of functions which are meromorphic for $|z|<1$. The proofs are based upon an examination of the properties of the functions $\varphi(z)$ defined by the relation $f[\varphi(z)]=f(z)$.

M. H. Heins (Princeton, N. J.).

Komatu, Yûsaku. Über einen Satz von Herrn Löwner. Proc. Imp. Acad. Tokyo **16**, 512–514 (1940). [MF 4326]

Let B denote the region in the w-plane consisting of $|w|<1$ slit along a Jordan arc L which lies in $|w|<1$ save for one endpoint on $|w|=1$ and omits the origin; and let $w=f(z)$ denote the normalized mapping function ($f(0)=0$; $f'(0)=e^{-t_0}, t_0>0$) which maps $|z|<1$ (1, 1) and conformally onto B. Loewner [Math. Ann. 89, 103–121 (1923)] has established that there exists a function $\kappa(t)$ continuous for $0\leqq t\leqq t_0$, $|\kappa(t)|=1$, such that $f(z)=f(z, t_0)$, where $f(z, t)$ satisfies the differential equation

$$\frac{\partial f(z,t)}{\partial t}=-f(z,t)\frac{1+\kappa(t)f(z,t)}{1-\kappa(t)f(z,t)},$$

and $f(z, 0)=z$. By means of the Poisson integral formula, the author gives a brief proof of this theorem under the hypothesis that L is rectifiable. *M. H. Heins.*

Komatu, Yûsaku. Zur konformen Schlitzabbildung. Proc. Imp. Acad. Tokyo **17**, 11–17 (1941). [MF 4333]

This paper is a continuation of the paper reviewed above. It studies geometric properties of the function $\kappa(t)$ of Loewner's theorem under the hypothesis that the slit L is an analytic arc. *M. H. Heins* (Princeton, N. J.).

Nevanlinna, Rolf. Ein Satz über offene Riemannsche Flächen. Ann. Acad. Sci. Fennicae (A) **54**, no. 3, 18 pp. (1940). [MF 3884]

Let F denote an open (abstract) Riemann surface, given in terms of an infinite sequence of subregions $K_1, K_2, \cdots, K_j, \cdots$, each of which is mapped conformally upon the unit circle. Let F_n be an increasing sequence of subregions filling up F, such that F_n is bounded by a finite system Γ_n of analytic arcs. Let ω_n be the harmonic function in F_n-F_0 which is equal to 1 on Γ_n and equal to zero on Γ_0. As n increases, ω_n decreases and approaches a harmonic function ω in $F-F_0$. If $\omega\equiv 0$, then the (ideal) boundary of F is said to be a zero boundary. Assuming that the above-mentioned system $K_1, K_2, \cdots, K_j, \cdots$ satisfies certain regularity conditions, the following sufficient condition for a zero boundary is derived. Let P_0 be a fixed point of F, and for each n let A_n be a point-set consisting of a finite number of the regions K_j, such that the following conditions are satisfied. (1) Each A_n separates P_0 from the (ideal) boundary of F. (2) There exists a finite constant N, such that no point of F belongs to more than N of the sets A_n. Let then l_n denote the number of the regions K_j that make up A_n. If the series $\sum(1/l_n)$ is divergent, then F has zero boundary. Several important applications (capacity of point-sets, type of a Riemann surface, generalization of the classical theory of Abelian integrals, automorphic functions) are briefly discussed or suggested. *T. Radó* (Columbus, Ohio).

Sugawara, Masao. On the general zetafuchsian functions. Proc. Imp. Acad. Tokyo **16**, 367–372 (1940). [MF 3479]

This paper is closely connected with a previous one [Ann. of Math. (2) **41**, 488–494 (1940); these Rev. **2**, 37]. The author studies generalizations of Poincaré's theta series and of the zeta series; by a generalization of Poincaré's methods, he proves their convergence for the most general fuchsian group. These important results will become still more important if it be possible to demonstrate that the series corresponding to a hyperfuchsian group cannot vanish identically. *G. Fubini* (Princeton, N. J.).

Selberg, Atle. Beweis eines Darstellungssatzes aus der Theorie der ganzen Modulformen. Arch. Math. Naturvid. **44**, 33–44 (1941). [MF 4423]

A new proof is given that every cusp-form of negative dimension $-k$ belonging to the principal congruence subgroup modulo Q can be expressed as a linear combination of Petersson functions

$$\sum(c\tau+d)^{-k}e^{2\pi i(\nu/Q)M\tau}$$

[cf. Petersson, Jber. Deutsch. Math. Verein. **49**, 49–75 (1939)]. Although the proof is carried out only for k an integer greater than 2, it will clearly go through for any real $k>2$. A further result is that, if $f(\tau)=\sum\alpha_\nu e^{2\pi i(\nu/Q)\tau}$ and $\varphi(\tau)=\sum\beta_\nu e^{2\pi i(\nu/Q)\tau}$ are two cusp forms of type $(-k, Q)$, then

$$\lim_{h\to 0} h\sum_{\nu=1}^{\infty}(\alpha_\nu\bar{\beta}_\nu/\nu^{k-1})e^{-2\pi(\nu/Q)h}$$

exists. The author defines the scalar product of f and φ to be this limit and notes that it differs from Petersson's scalar product only by a constant factor.

H. S. Zuckerman (Seattle, Wash.).

Fubini, Guido. On hyperautomorphic functions. Univ. Nac. Tucumán. Revista A. **1**, 87–94 (1940). [MF 4056]

Fubini, Guido. Equazioni differenziali per i periodi di un integrale iperellittico. Univ. Nac. Tucumán. Revista A. **1**, 73–79 (1940). [MF 4055]

The author points out a simple method for finding differential equations satisfied by the periods

$$w=\int_\lambda(u-a_1)^{\sigma_1}\cdots(u-a_n)^{\sigma_n}du.$$

The equations include both partial differential equations, which can be reduced to the first order, and ordinary differential equations in which only one of the a_i is taken as variable. *H. S. Zuckerman* (Seattle, Wash.).

Walsh, J. L. and Sewell, W. E. On the degree of polynomial approximation to analytic functions: problem β. Trans. Amer. Math. Soc. **49**, 229–257 (1941). [MF 3923]

Let $f(z)$ be regular in $|z|<1$. The authors define the class $L(p, \alpha)$, p integer, $0<\alpha\leqq 1$, by the following conditions according as $p\geqq 0$ or $p<0$: (a) $f^{(p)}(z)$ satisfies a Lipschitz condition of order α; (b) $|f(re^{i\theta})|<M(1-r)^{p+\alpha}$, $r<1$. With this terminology it is shown that, if $f(z)\in L(p, \alpha)$ and $\rho>1$, polynomials $p_n(z)$ of degree n exist such that $|f(z)-p_n(z)|<M\rho^{-n}n^{-p-\alpha}$, $|z|=\rho^{-1}$. Conversely, if for each n a polynomial $p_n(z)$ of degree n exists such that $|f(z)-p_n(z)|<M\rho^{-n}n^{-p-\alpha-1}$, $|z|=\rho^{-1}$, p integral, $0<\alpha\leqq 1$, then $f(z)\in L(p, \alpha)$ if $p+\alpha+1$ is not a positive integer and $f^{(p+2)}(z)\in L(-2, 1)$ if $p+\alpha+1$ is a positive integer (so that $p\geqq -1$, $\alpha=1$). These theorems are extended by means of a conformal mapping to domains bounded by analytic Jordan curves. In addition, the case of a line segment is considered in particular, and approximations measured by an integral are investigated. Finally similar approximation theorems are proved for the class $L_2(p, \alpha)$, p integer,

$0<\alpha\leqq 1$, defined by the following conditions according as $p<0$, $p=0$, $p>0$:

(a) $$\left\{\int_{-\pi}^{+\pi}|f(re^{i\theta})|^2 d\theta\right\}^{\frac{1}{2}}<M(1-r)^{p+\alpha};$$

(b) $$\int_{-\pi}^{+\pi}|f(e^{i(\theta+h)})-f(e^{i\theta})|^2 d\theta<Mh^{2\alpha};$$

(c) $$f^{(p+1)}(z)\in L_2(-1,\alpha).$$

G. Szegö (Stanford University, Calif.).

Gelbart, Abe. On the growth properties of a function of two complex variables given by its power series expansion. Trans. Amer. Math. Soc. **49**, 199–210 (1941). [MF 3921]

The author considers a function $f(z_1, z_2)$ regular in the closure $\bar{M}$ of a finite four-dimensional region M which is bounded by two three-dimensional hypersurfaces S_1 and S_2, where S_1 and S_2 are of a fairly general character. The first theorem gives a lower bound for $\max|f(z_1, z_2)|$ on the intersection $G=S_1\cap S_2$, the lower bound being in terms of the coefficients of the power-series for f and certain quantities depending only upon the surfaces S_1 and S_2. The proof is by means of an iterated Cauchy integral as developed by Bergman. On applying this lower bound to the function $F(z_1, z_2)=\exp\{f(z_1, z_2)e^{-ia}\}$, the author obtains a lower bound for the image of the two dimensional surface $G=S_1\cap S_2$ on the f_1, f_2-plane (where $f(z_1, z_2)=f_1+if_2$). A result similar to that of the first theorem is obtained for more general regions (bounded by three hypersurfaces) and finally an inequality is derived for the function f on a portion of the boundary of M not on G.

W. T. Martin (Princeton, N. J.).

Bergman, Stefan. On the surface integrals of functions of two complex variables. Amer. J. Math. **63**, 295–318 (1941). [MF 4152]

In earlier papers the author has considered a general class of domains $\mathfrak{M}$ possessing distinguished boundary surfaces S, and has given an integral formula for those domains representing an arbitrary analytic function of two complex variables, regular in $\overline{\mathfrak{M}}$, by means of an integral over S. In the present paper the author considers the surface integral of a meromorphic function f taken over the distinguished surface S, and shows that this integral is related in a definite manner to certain expressions which are associated with special points in the domain $\mathfrak{M}$ and on the singularity surface of the function f. The method of attack is based upon the notion of an extended class of functions introduced by the author in an earlier paper in connection with the Dirichlet problem. As the author indicates, certain of the surface integrals considered are related to those introduced by Poincaré. *W. T. Martin* (Princeton, N. J.).

Nisigaki, Hisami. Zur Theorie der Quaternionenfunktion. II. Tôhoku Math. J. **47**, 217–236 (1940). [MF 4023]

Let $Y=\sum_1^4 y_ie_i$ be a function of the quaternion $X=\sum_1^4 x_ie_i$, where the e_i are units. The author calls $Y=F(X)$ analytic in a region B if the y_i are analytic functions of the (real or complex) variables $x_1, \cdots, x_4$ and if the differential $dY=\sum_1^4 dy_ie_i$ is representable in the form $dY=\sum_1^4\varphi_{ij}e_i dX e_j$, where the φ_{ij} are analytic functions of $x_1, \cdots, x_4$. He says that Y belongs to class $C_\rho(1\leqq\rho\leqq 4)$ in B if Y is analytic and if at every point of B the differential is expressible as $dY=\sum_1^\rho U_i dX V_i$, where $U_1, \cdots, U_\rho$, $V_1, \cdots, V_\rho$ are linearly independent functions. In a preceding paper [Tôhoku Math. J. **45**, 73–102 (1938)] the author treated the class C_1; in the present paper he proves several general theorems for class C_2, including some results on series, some on one-to-one mappings and on the Jacobian of an inner mapping of a bounded region.

W. T. Martin (Princeton, N. J.).

Theory of Series

Rado, R. The distributive law for products of infinite series. Quart. J. Math., Oxford Ser. **11**, 229–242 (1940). [MF 3785]

The problem solved is that of characterizing sequences (m_1, n_1), (m_2, n_2), (m_3, n_3), $\cdots$ of pairs of positive integers such that the series in the left member of

$$\sum_{k=1}^{\infty}X_{m_k}Y_{n_k}=\left(\sum_{m=1}^{\infty}X_m\right)\left(\sum_{n=1}^{\infty}Y_n\right)$$

is convergent and equal to the right member whenever the two series in the right member both converge; such a sequence is called distributive. For each $K=1, 2, 3, \cdots$, let R_K be the minimum number of rectangles which collectively contain the points $(m_1, n_1), \cdots, (m_K, n_K)$ and no other points of the integer lattice. A sequence (m_k, n_k) is distributive if and only if it contains each pair of positive integers exactly once and its sequence $R_1, R_2, R_3, \cdots$ is bounded. *R. P. Agnew* (Ithaca, N. Y.).

Perlin, I. E. Series with deleted terms. Amer. Math. Monthly **48**, 93–97 (1941). [MF 3911]

Sufficient conditions are obtained in order that an infinite series $\sum a_n$ shall converge after some of the terms have been deleted. The proof is based on Abel's transformation. As an application, that series is convergent which is obtained after deleting from the harmonic series $\sum 1/n$ those terms for which two consecutive digits of n are alike.

A. C. Schaeffer (Stanford University, Calif.).

Rajagopal, C. T. On Abel's divergence test for series of positive terms. Math. Student **8**, 118–123 (1940). [MF 4314]

In this paper several theorems on infinite series with positive terms are proved and integral analogues are stated without proof. There are numerous corollaries. The first theorem is a direct generalization of Abel's theorem: If $a_n\downarrow 0$ as $n\to\infty$ the condition $\lim na_n=0$ is necessary for the convergence of $\sum a_n$. The second theorem proved seems to the reviewer to be of most interest. A part of this theorem is the following: If $0=D_0<D_1<D_2<\cdots<D_n\to\infty$ and $d_n=D_n-D_{n-1}$ and if $a_n/d_n\downarrow 0$ and $\sum_1^\infty a_n$ is convergent, then

$$\sum_1^{\infty}D_n\left(\frac{a_n}{d_n}-\frac{a_{n+1}}{d_{n+1}}\right)$$

converges to the same value as $\sum a_n$. If one series converges, then so does the other.

T. Fort (Bethlehem, Pa.).

Rajagopal, C. T. Remarks on some generalizations of Cauchy's condensation and integral tests. Amer. Math. Monthly **48**, 180–185 (1941). [MF 4133]

The first paragraph of this paper is as follows: "This

paper is a collection of theorems which may be placed in the same context as the two in Hukam Chand's article, 'On some generalizations of Cauchy's condensation and integral tests,' this Monthly, vol. 46, 1939, pp. 338–341 [cf. these Rev. 1, 10]. The two theorems in question are contained in de la Vallée-Poussin's 'Cours d'Analyse' (vol. 1, 1926, pp. 398–399.)." The principal contribution is a none-too-great generalization of the Weierstrass ratio test for series of complex terms. *T. Fort* (Bethlehem, Pa.).

Hardy, G. H. Note on a divergent series. Proc. Cambridge Philos. Soc. **37**, 1–8 (1941). [MF 3765]

The author considers the divergent series $\sum_0^\infty n!z^n$ in the plane P cut along the positive z axis. As is well known, this series can be regarded as the development of

$$G(z)=\int_0^\infty \frac{e^{-t}}{1-zt}dt.$$

The sector $0\leqq r\leqq R$, $0<\eta\leqq\theta\leqq 2\pi-\eta<2\pi$ is denoted by $P(R,\eta)$. It is then shown that the more obvious methods of summation fail to sum $G(z)$. The author extends the notion of Lindelöf summation by introducing $\lambda_0=\lambda_1=\lambda_2=0$, $\lambda_n=n\log n\log\log n$, $n>3$, and denotes a series $\sum c_n$ as summable (Λ) if

$$\lim_{\delta\to 0}\sum e^{-\delta\lambda_n}c_n=s.$$

It is then proved that $\sum n!z^n$ is summable (Λ) to $G(z)$ throughout P and uniformly so in any $P(R,\eta)$. It is further shown that the series obtained by replacing $1/(1-zt)$ by a more general function $f(z,t)$ in the definition of $G(z)$ are also summable. If $f(z)=\sum_0^\infty a_nz^n$, then these series are $\sum_0^\infty a_nn!z^n$. The condition on $f(z)$ for summability (Λ) is that $f(z)$ be regular at the origin and in P, and $f(z)=O(|z|^k)$ for large $|z|$ and some k uniformly in $P(\infty,\eta)$.

N. Levinson (Cambridge, Mass.).

Hill, J. D. and Hamilton, H. J. Operation theory and multiple sequence transformations. Duke Math. J. **8**, 154–162 (1941). [MF 3950]

Proofs of theorems on multiple sequence transformations, previously given by Hamilton [Duke Math. J. **2**, 29–60 (1936); **4**, 341–358 (1938); **5**, 293–297 (1939)], are phrased in terms of the Banach theory of linear operations.

R. P. Agnew (Ithaca, N. Y.).

Gavurin, M. K. Sur les séries potentielles abstraites. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 9–11 (1940). [MF 3695]

The author continues his study on abstract Taylor series. [C. R. (Doklady) Acad. Sci. URSS (N.S.) **22**, 548–552 (1939)]. He proves for these series theorems analogous to those in the classical theory (successive derivation, expansion in the neighborhood of a point contained in the "circle" of convergence of a given series, and so forth).

S. Mandelbrojt (Houston, Texas).

Garabedian, H. L., Hille, Einar and Wall, H. S. Formulations of the Hausdorff inclusion problem. Duke Math. J. **8**, 193–213 (1941). [MF 3955]

A sequence $\{x_n\}$ is regular if there exists a regular mass function $\phi_x(u)$ such that $x_n=\int_0^1u^nd\phi_x(u)$; $\phi_x(u)$ is of bounded variation on $[0,1]$, is zero if $u\leqq 0$, one if $u\geqq 1$ and $\frac{1}{2}[\phi_x(u+0)+\phi_x(u-0)]$ if $0\leqq u<1$. With each regular sequence there is associated a regular moment function $x(z)=\int_0^1u^zd\phi_x(u)$ and a regular moment generating function

$$f_x(t)=x_0-x_1t+x_2t^2-\cdots=\int_0^1 d\phi_x(u)/(1+tu).$$

With each regular sequence there is associated a regular Hausdorff summation matrix denoted by $\{H,\phi_x(u)\}$. Let $\{a_n\}$ and $\{b_n\}$, $b_n\neq 0$, be two regular sequences. It was proved by Hausdorff that a necessary and sufficient condition in order that $\{H,\phi_a(u)\}\supset\{H,\phi_b(u)\}$ is that $a_n=b_nc_n$, $n=0,1,2,\cdots$, where $\{c_n\}$ is a regular sequence. Two necessary and sufficient conditions for this were stated (without proof) by Hille and Tamarkin [Proc. Nat. Acad. Sci. U. S. A. **20**, 140–144 (1934)]. The authors make more precise the second of these conditions, and also prove a third necessary and sufficient condition: There exists a regular mass function $\phi_c(u)$ such that $f_a(t)=\int_0^1f_b(tu)d\phi_c(u)$, $|t|<1$. In the remaining part of the paper the authors derive necessary and sufficient conditions for $[H,\phi(u)]\supset(C,n)$, $n\geqq 1$ an integer, and discuss various inclusion problems associated with Hölder and Cesàro methods.

J. D. Tamarkin (Providence, R. I.).

Durañona y Vedia, A. Abelian and Tauberian theorems for two variables. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie 2: Revista **4**, 291–324 (1940). (Spanish. French summary) [MF 4199]

Let $\sum a_{mn}$ be a double series of real terms; let $\{\sigma_{mn}\}$ denote that C_1 transform of the sequence $\{s_{mn}\}$ of its partial sums; let $t_{mn}=\sum_{j=0}^m a_{mj}$, $t'_{mn}=\sum_{i=0}^n a_{in}$. The author's Tauberian results are for the most part parallel to results of Knopp [Math. Z. **45**, 573–589 (1939); these Rev. **1**, 51], but while Knopp usually deduces convergence from summability plus a Tauberian condition, the author deduces bounded convergence from bounded summability plus a Tauberian condition. In particular, the author shows for C_1 summability that $\sigma_{mn}\to s$ boundedly and (*) $mt_{mn}<K$, $nt'_{mn}<K$ together imply that $s_{mn}\to s$ boundedly. He gives an example to show that (*) is more general than Knopp's condition $(m^2+n^2)a_{mn}<K$, contrary to the (apparently inadvertent) assertion made by Meyer-König [Math. Z. **46**, 157–160 (1940); these Rev. **1**, 219]. Condition (*) had also been used by Fujiwara, as the author points out. The author generalizes Knopp's Tauberian theorems for Abel summability by using (*) and also other Tauberian conditions. [The author's Theorem XI is essentially the same as Knopp's Theorem 7, contrary to his apparent belief, since bounded Abel summability is deducible from Knopp's hypotheses.] The author's very compact notation makes some of his proofs quite short.

R. P. Boas, Jr. (Durham, N. C.).

Karamata, J. Einige Sätze über iterierte Mittelbildungen. Proc. Benares Math. Soc. **1**, 15–24 (1939). [MF 1527]

Let $P_n=p_1+p_2+\cdots+p_n>0$ for each $n=1,2,\cdots,k$. A function $s(x)$ is called summable $M(p_1,p_2,\cdots,p_k)$ to s if $S_k(x)\to s$ as $x\to\infty$ when

$$S_k(x)=S\{p_1,p_2,\cdots,p_k;s(x)\}$$
$$=\frac{P_1P_2\cdots P_k}{x^{P_k}}\int_0^x t_k^{p_k-1}dt_k\int_0^{t_k}\cdots\int_0^{t_3}t_2^{p_2-1}dt_2\int_0^{t_2}t_1^{p_1-1}s(t_1)dt_1.$$

If a function $s(x)$ is summable to s by some one of these methods of order k, then $s(x)$ is summable to s by each of the methods of order k' provided only $k'\geqq k$. Two of these methods of summability having equal orders are equivalent.

Thus the Hölder method $H_k \equiv M(1, 0, \cdots, 0)$ is equivalent to the Cesàro method $C_k \equiv M(1, 1, \cdots, 1)$. An infinite sequence $p_1, p_2, \cdots$ determines a method $M(p_1, p_2, \cdots)$ of infinite order by which a function $s(x)$ is summable to s if the superior and inferior limits $\bar{S}_n$ and $\underline{S}_n$ as $n \to \infty$ of $S\{p_1, \cdots, p_n; s(x)\}$ are such that $\bar{S}_n \to s$ and $\underline{S}_n \to s$ as $n \to \infty$. When a is real and not 0, the function x^{ia} is summable by the Hölder method $H_\infty \equiv M(1, 0, 0, \cdots)$ to 0 but is not summable by the Cesàro method $C_\infty \equiv M(1, 1, \cdots)$. The remainder of the paper is devoted to Tauberian oscillation theorems for methods of finite and infinite order.

R. P. Agnew (Ithaca, N. Y.).

Agnew, Ralph Palmer. Some remarks on a paper entitled "General Tauberian theorems." J. London Math. Soc. **15**, 242–246 (1940). [MF 4033]

The author points out that the transition from Tauberian oscillation theorems to Tauberian convergence theorems as made by Pitt [Proc. London Math. Soc. (2) **44**, 243–288 (1938)] contains a slip the effect of which is described in the following review. *N. Levinson* (Cambridge, Mass.).

Pitt, H. R. Note on the preceding paper. J. London Math. Soc. **15**, 247 (1940). [MF 4034]

The author points out that the major results of "General Tauberian Theorems," being on oscillation theorems, are still valid. The Tauberian theorems 14, 15, 16 and 17 of his paper, on the basis of the remarks of Agnew, will be correct if A (or C) in these theorems is taken as zero. Thus in each hypothesis instead of $s(x)$ being summable to a constant A, in each case, $s(x)$ should be summable to zero. The Tauberian conclusion in each case would be $s(x) \to 0$, instead of A. *N. Levinson* (Cambridge, Mass.).

Fourier Series and Integrals, Theory of Approximation

Friedman, Bernard. Fourier coefficients of bounded functions. Bull. Amer. Math. Soc. **47**, 84–92 (1941). [MF 3816]

It is shown that a necessary and sufficient condition that $2n+1$ real numbers (*) $a_0, a_1, b_1, \cdots, a_n, b_n$ should be the first $2n+1$ Fourier coefficients of a function f, periodic and not exceeding 1 in absolute value, is that (*), treated as a point in $2n+1$ dimensional Euclidean space, should belong to a certain convex body B_{2n+1} of that space. The boundary of B_{2n+1} corresponds to functions $f(x) = \text{sign } T_n(x)$, where T_n is an arbitrary trigonometric polynomial of order n. For every function $f(x)$ periodic and not exceeding 1 in absolute value, and to every integer $n > 0$, there exists a step function assuming only the values ± 1 and having the same first $2n+1$ Fourier coefficients as $f(x)$.

A. Zygmund (South Hadley, Mass.).

Pinkewitch, W. Sur l'ordre du reste de la série de Fourier des fonctions dérivables au sens de Weyl. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 521–528 (1940). (Russian. French summary) [MF 4089]

Let H^α denote the class of functions of period 2π, differentiable α times and satisfying the inequality $|f^{(\alpha)}(x)| \leqq K$, where K is a constant. Let

$$C_n^{(\alpha)} = \sup_{f \in H^\alpha} |f(x) - s_n(x, f)|,$$

where $s_n(x, f)$ denotes the nth partial sum of the Fourier series of $f(x)$. Kolmogoroff has shown [Ann. of Math. (2) **36**, 521–526 (1935)] that

$$C_n^{(\alpha)} = \frac{4K \log n}{\pi^2 n^\alpha} + O\left(\frac{1}{n^\alpha}\right)$$

for every integer $\alpha \geqq 1$. The author now shows that the theorem holds for any fractional $\alpha > 0$, provided we mean by $f^\alpha(x)$ Weyl's fractional derivative of $f(x)$ of order α.

A. Zygmund (South Hadley, Mass.).

Nikolski, S. Sur l'allure asymptotique du reste dans l'approximation au moyen des sommes de Fejér des fonctions vérifiant la condition de Lipschitz. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 501–508 (1940). (Russian. French summary) [MF 4087]

Let $f(x)$ be a function of period 2π satisfying a Lipschitz condition of order α, $0 < \alpha < 1$, and let $\sigma_n(f, x)$ denote the Fejér means of the Fourier series of $f(x)$. It is well known [S. Bernstein, Acad. Roy. Belgique. Bull. Cl. Sci. **4**, 1–104 (1912)] that $f(x) - \sigma_n(f, x) = O(n^{-\alpha})$. The author now shows that, if f is of period 2π and satisfies the condition

$$|f(x'') - f(x')| \leqq K|x'' - x'|^\alpha$$

for all x', x'', $0 < \alpha < 1$, then $|\sigma_n - f|$ never exceeds

$$K \cdot 2\Gamma(\alpha)/(\pi(1-\alpha)) \sin \alpha\pi/2 \cdot n^{-\alpha} + o(n^{-\alpha}),$$

and the coefficient of $n^{-\alpha}$ cannot be replaced by anything smaller. A similar extension is given to another theorem of Bernstein concerning the case $\alpha = 1$. The results are extended to the interpolating polynomials of Jackson.

A. Zygmund (South Hadley, Mass.).

Nikolski, S. Sur certaines méthodes d'approximation au moyen de sommes trigonométriques. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 509–520 (1940). (Russian. French summary) [MF 4088]

Let $s_n(x)$ denote the partial sums of the Fourier series of a continuous function $f(x)$, and let

$$\sigma_{n,p}(x) = (s_{n-p}(x) + s_{n-p+1}(x) + \cdots + s_n(x))/(p+1).$$

Since

$$\sigma_{n,p} = \{(n+1)\sigma_n - (n-p)\sigma_{n-p-1}\}/p,$$

where σ_n denotes the first Cesàro means of the series considered, it is plain that $\sigma_{n,p_n}(x)$ tends uniformly to $f(x)$ as $n \to \infty$, if only (*) $\liminf p_n/n > 0$. The author quotes an unpublished result of Verbitzky to the effect that the condition (*) is also necessary if $\sigma_{n,p_n}(x) - f(x)$ is to tend uniformly to 0 for every continuous f, and shows that the maximum of $|\sigma_{n,p}|$ for all x and all functions f not exceeding 1 in absolute value is

$$\frac{4}{\pi^2} \log \frac{n}{p+1} + O(1).$$

An analogous result is obtained for trigonometric interpolation in the case of equidistant nodal points.

A. Zygmund (South Hadley, Mass.).

Maruyama, Gisirô. Summability of Fourier series. Tôhoku Math. J. **47**, 255–260 (1940). [MF 4026]

Let $s_n(x)$ denote the nth partial sum of the Fourier series of an L-integrable function $f(x)$. It is known that the sequence (*) $s_{p_1}(x), s_{p_2}(x), \cdots$ is summable $(C, 1)$ almost

everywhere if, for example, $p_\nu=\nu^k$, where k is any positive integer [Zalcwasser, Studia Math. 6, 82–88 (1936); Izumi and Kawata, Tôhoku Math. J. 46, 154–158 (1939); these Rev. 1, 225]. These sequences are convex. The author now shows that the sequence (*) is summable $(C, 1)$ almost everywhere if $p_\nu=[q_\nu]$, where $\Delta^2 q_\nu \leqq 0$ and $q_\nu/(q_\nu-q_{\nu-1})=O(\nu)$.

A. Zygmund (South Hadley, Mass.).

Chow, H. C. Cesàro means connected with the allied series of a Fourier series. J. Chinese Math. Soc. **2**, 291–300 (1940). [MF 4214]

It is known that, if $\sum c_n \sin nt$ is the Fourier series of a function $\psi(t)$ of bounded variation, the sequence nc_n converges (C, α) to $(2/\pi)\psi(+0)$ for $\alpha>0$. [See Zygmund, Trigonometrical Series, p. 62.] The present paper is devoted to some generalizations of this theorem, based on the behavior of the function

$$\psi_\alpha(t)=\alpha t^{-\alpha}\int_0^t (t-u)^{\alpha-1}\psi(u)du.$$

For example: If $\alpha \geqq 1$ and $\psi_\alpha(t)$ is of bounded variation in $(0, \eta)$, $\eta>0$, the sequence nc_n converges (C, α) to $(2/\pi)\psi_\alpha(+0)$; if $0<\alpha<1$ and if $\psi_\alpha(t)$ is of bounded variation in $(0, \pi)$, the sequence nc_n converges (C, β) to $(2/\pi)\psi_\alpha(+0)$ for $\beta>\alpha$.

R. Salem (Montreal, Que.).

Grünwald, Géza. Über die Summabilität der Fourierschen Reihe. Acta Univ. Szeged. Sect. Sci. Math. **10**, 55–63 (1941). [MF 4432]

The chief result of the paper is the following theorem: (A) For every function $f(x, y)$ integrable L in the square $0 \leqq x<2\pi$, $0 \leqq y \leqq 2\pi$, and of period 2π with respect to x, y,

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} s_{k,k}(x, y; f)=f(x, y)$$

at almost every point x, y. Here $s_{n,n}(x, y; f)$ denotes the nth quadratic partial sum of the (double) Fourier series of $f(x, y)$. Taking $f(x, y)=g(x)h(y)$, where f and g are periodic and integrable functions of a single variable, the author obtains as a corollary that: (B) For any pair of integrable functions g and f

$$(*) \qquad \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} s_k(x; g)s_k(y; h)=g(x)h(y)$$

at almost every point x, y of the square $0 \leqq x \leqq 2\pi$, $0 \leqq y \leqq 2\pi$ ($s_n(x; g)$ denotes the nth partial sum of the Fourier series of $g(x)$). It may be added that the corollary (B) also follows, even in a stronger form, from a theorem of Marcinkiewicz [J. London Math. Soc. **14**, 162–168 (1939)]. Marcinkiewicz showed that for any integrable function $g(x)$

$$(**) \qquad \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=0}^{n} [s_k(x; g)-g(x)]^2=0$$

almost everywhere in $(0, 2\pi)$ and so, by an application of Schwarz's inequality, (*) is valid except when x and y belong to certain linear sets of measure zero. [The exponent 2 in (**) may be replaced by any number $q>0$; see a paper of the reviewer, which is to appear in the Proc. London Math. Soc.]

Theorem A was also obtained independently (for Nörlund's means, and in particular for the $(C, \alpha)(\alpha>0)$ means of the sequence $s_{n,n}(x, y; f)$) by J. Herriot, whose paper is to appear shortly in the Trans. Amer. Math. Soc.

A. Zygmund (South Hadley, Mass.).

Cesari, Lamberto. Sulle funzioni di più variabili generalmente a variazione limitata e sulla convergenza delle relative serie multiple di Fourier. Pont. Acad. Sci. Comment. **3**, 171–197 (1939). [MF 4101]

The paper is devoted to the extension to functions of three variables of tests previously obtained by the author for the convergence of Fourier series of functions of two variables [cf. Rend. Sem. Mat. Univ. Roma (4) **1**, 277–294 (1937); Atti 1° Congresso Naz. Unione Mat. Italiana, Firenze, 1937, pp. 215–218]. The author points out that the proofs given by him for functions of two variables have been simplified by Tonelli [Ann. Scuola Norm. Super. Pisa (2) **6**, 315–326 (1937)] whose argument can be applied to the case of three variables as well.

R. Salem.

Kawata, Tatsuo. The Fourier series of the characteristic function of a probability distribution. Tôhoku Math. J. **47**, 121–125 (1940). [MF 4013]

Let $\varphi(t)$ be the characteristic function of a symmetric distribution function $V(x)$, that is,

$$\varphi(t)=\int_{-\infty}^{+\infty} e^{itx}dV(x),$$

and let $f(t)$ be the function coinciding with $\varphi(t)$ in the interval $(-R, +R)$ and of period $2R$. Then the Fourier series of $f(t)$ is absolutely convergent, provided that

$$\int_1^\infty \log t\, dN(t)<\infty.$$

In this condition, $\log t$ cannot be replaced by $(\log t)^{1-\epsilon}$, for any $\epsilon>0$.

A. Zygmund (South Hadley, Mass.).

Haviland, E. K. On the distribution functions of the reciprocal of a function and of a function reduced mod 1. Amer. J. Math. **63**, 408–414 (1941). [MF 4160]

The following theorems are proved: (I) If $x(t)$ possesses an asymptotic distribution function $\sigma(\xi)$, then the function $y(t)=1/x(t)$ ($x(t)\neq 0$, and $y(t)$ arbitrary if $x(t)=0$) possesses an asymptotic distribution function $\tau(\xi)$ if and only if $\sigma(\xi)$ is continuous at $\xi=0$. Furthermore, $\tau(\xi)=1+\sigma(0)-\sigma(\xi^{-1})$ for $\xi \geqq 0$ and $\tau(\xi)=\sigma(0)-\sigma(\xi^{-1})$ for $\xi \leqq 0$. (II) If $x(t)$ possesses an asymptotic distribution function $\sigma(\xi)$, then also $x(t)-[x(t)]$ possesses an asymptotic distribution function $\rho(\xi)$, and

$$\rho(\xi)=\sum_{n=-\infty}^{\infty}\{\sigma(\xi+n)-\sigma(n)\}.$$

M. Kac (Ithaca, N. Y.).

Ríos, Sixto. On the problem of the overconvergence of Dirichlet series whose sequences of exponents have unbounded maximum density. Revista Acad. Ci. Madrid **34**, 163–179 (1940). (Spanish) [MF 3707]

The author furnishes a partial answer to a question proposed by Vl. Bernstein [Leçons sur les séries de Dirichlet, Paris, 1933, pp. 193–194]. He gives examples of Dirichlet series (when the maximum density of the exponents λ_n is infinity) (1) which fail to overconverge in any part of the half-plane of holomorphism, (2) which overconverge in the entire half-plane of holomorphism, (3) which overconverge in a part of the half-plane of holomorphism. He proves some theorems on the axis of holomorphism and on that of overconvergence.

S. Mandelbrojt (Houston, Texas).

Smith, C. V. L. The fractional derivative of a Laplace integral. Duke Math. J. 8, 47–77 (1941). [MF 3945]

The author studies the integral

$$f(x)=\int_0^\infty e^{-xt}t^\rho d\alpha(t), \qquad \rho>0, \tag{1}$$

where $\alpha(t)$ is of bounded variation in every finite interval, $\alpha(0)=\alpha(0+)=0$ and $\alpha(t)=\frac{1}{2}[\alpha(t+)+\alpha(t-)]$ for $t>0$. With a suitable definition of fractional differentiation, (1) is, except for sign, the ρth derivative of an ordinary Laplace-Stieltjes integral [(1) with $\rho=0$]. The integral (1) is inverted by

$$\alpha(t)=\lim_{k\to\infty}\int_0^t u^{-\rho}L_{k,u}[f(x)]du,$$

where $L_{k,u}[f]$ is the Post-Widder inversion operator for the Laplace integral:

$$k!L_{k,t}[f(x)]=(-1)^k(k/t)^{k+1}f^{(k)}(k/t)$$

[see Widder, Trans. Amer. Math. Soc. 39, 244–298 (1936)]. Representation of a given function $f(x)$ by integrals of the form (1) is studied for $\alpha(t)$ in various classes. A particularly interesting result is that $f(x)$ is represented by (1) with $\alpha(t)$ non-decreasing and bounded if and only if $f(x)$ is completely monotonic in $(0, \infty)$, $f(\infty)=0$, and

$$\int_0^\infty x^{\rho-1}f(x)dx<\infty; \tag{2}$$

if $\alpha(t)$ is not required to be bounded, the lower limit of the integral in (2) is replaced by 1. *R. P. Boas, Jr.*

Pollard, Harry. Real inversion formulas for Laplace integrals. Duke Math. J. 7, 445–452 (1940). [MF 3412]

The author continues his work [Duke Math. J. 6, 420–424 (1940); these Rev. 1, 331] on the inversion of the convergent Laplace integrals

$$f(x)=\int_0^\infty e^{-xt}d\alpha(t), \tag{1}$$

$$f(x)=\int_0^\infty e^{-xt}\phi(t)dt, \qquad x>c, \tag{2}$$

where $\alpha(t)$ and $\phi(t)$ are, respectively, of bounded variation and Lebesgue integrable on finite intervals. A number of inversion formulae for both (1) and (2) are obtained which are derived from the following chief result. Let $\{\theta_k(t)\}$ be a sequence of real functions such that $\theta_k/k\to 0$ as $k\to\infty$, for each $t>0$ but not necessarily uniformly in t. Then (2) implies

$$\lim_{k\to\infty}\left(\frac{k+\theta_k}{t}\right)^{k+1}\frac{(-1)^k}{k!}f^{(k)}\left(\frac{k+\theta_k}{t}\right)=\phi(t) \tag{3}$$

for all $t(>0)$ of the Lebesgue set of $\phi(t)$. If $\phi(t+)$, $\phi(t-)$ exist at $t(>0)$, and if $\theta_k/k^{\frac{1}{2}}\to 0$ as $k\to\infty$, then the limit (3) equals $\frac{1}{2}(\phi(t+)+\phi(t-))$. *I. J. Schoenberg.*

Hedge, L. B. Moment problem for a bounded region. Bull. Amer. Math. Soc. 47, 282–285 (1941). [MF 4176]

The moment problem under consideration is the establishing of necessary and sufficient conditions for a sequence $\{\mu_{m_1,m_2,\cdots,m_n}\}$ to be representable in the form

$$\mu_{m_1,m_2,\cdots,m_n}=\int_E x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}d\Phi(E),$$

where $n\geqq 1$; E is a given closed bounded set in Euclidean n-space; and Φ is a completely additive set function defined at least over Borel sets, and restricted to belong to one of various classes (for example, the class of indefinite integrals of functions of L^p, or the class of functions satisfying $\Phi(B)\geqq 0$ for all Borel sets B). When $n=1$ and E is a unit interval, the problem reduces to Hausdorff's moment problem [Math. Z. 16, 220–248 (1923)]; when $n=2$ and E is the circumference of the unit circle, the problem becomes the coefficient problem for Fourier series. The author's conditions generalize one of the sets given by Hausdorff; they require the existence of a set of polynomials biorthogonal over E and of a regular Toeplitz transformation under which the set of polynomials has certain convergence and positivity properties. For the unit sphere in n-space, suitable polynomials exist. *R. P. Boas, Jr.* (Durham, N. C.).

Boas, R. P., Jr. A general moment problem. Amer. J. Math. 63, 361–370 (1941). [MF 4157]

Let (R_p) be the statement: $\{\sum_1^\infty |a_n|^p\}^{1/p}<\infty$ implies that there is a function $f(x)$ such that $a_n=\int f(x)\phi_n(x)dx$ and

$$(\int|f(x)|^{p'}dx)^{1/p'}\leqq M(\sum|a_n|^p)^{1/p},$$

and (B_p) the statement: The equations $a_n=\int f(x)\phi_n(x)dx$, $n=1, 2, \cdots$, imply that

$$(\sum|a_n|^{p'})^{1/p'}\leqq N(\int|f(x)|^p dx)^{1/p}.$$

Here integration is taken over $(0, 1)$, $1/p+1/p'=1$, $\phi_n\epsilon L_2$ and $L_{p'}$ in case of (B_p) and $\phi_n\epsilon L_p$ in case of (R_p). Finally let (ρ_p) and (β_p) be, respectively, the statements:

$$(\int|\sum c_n\phi_n(x)|^p dx)^{1/p}\geqq\mu(\sum|c_n|^{p'})^{1/p'},$$
$$(\int|\sum c_n\phi_n(x)|^{p'}dx)^{1/p'}\leqq\nu(\sum|c_n|^p)^{1/p},$$

where $1\leqq p\leqq\infty$, μ and ν depend only on $\{\phi_n\}$ and summation is extended over any finite set of $\{c_n\}$. The author proves the following results. (1) If $1\leqq p<\infty$ then (R_p) is equivalent to (ρ_p); if $1\leqq p\leqq\infty$ then (B_p) is equivalent to (β_p). (2) If $1\leqq p<\infty$ then (A_p) and (β_p) together imply (ρ_p). Here (A_p) is the statement

$$|\sum_{m\neq n}c_m d_n\int\phi_m(x)\bar{\phi}_n(x)dx|\leqq\alpha(\sum|c_m|^p)^{1/p}(\sum|d_n|^{p'})^{1/p'}, \qquad 0<\alpha<1,$$

for all finite sets $\{c_m\}$, $\{d_n\}$. (3) If $1<p\leqq 2$ and $|\lambda_n-n|\leqq\delta\leqq(p-1)/(48+36\pi)$, $n=0, \pm1, \pm2, \cdots$, then $\{e^{2\pi i\lambda_n x}\}$ has (R_p). It is mentioned that the last result was also obtained by N. Wiener. *J. D. Tamarkin.*

Menchoff, D. Sur la sommation des séries de fonctions orthogonales par des méthodes de Cesàro. Rec. Math. [Mat. Sbornik] N.S. 8(50), 121–136 (1940). (French. Russian summary) [MF 3194]

It is shown that an arbitrary sequence of orthonormal functions on an interval (a, b) can be rearranged so that every series formed from the rearranged sequence is summable almost everywhere by Cesàro means of arbitrary positive order, if the sum of the squares of the coefficients is convergent. This improves an earlier result by the same author, which was to the same effect except that it now appears for the first time that a single rearrangement will serve for all positive orders of summability. The theorem is obtained as corollary of a still more general one relating to summability in terms of a sequence of Toeplitz matrices. *D. Jackson* (Minneapolis, Minn.).

Markoff, A. Functions generated by developing power series in continued fractions. Duke Math. J. **7**, 85–96 (1940). [MF 3391]

Originally published in the appendix to Memoirs (Zapiski) of the Imperial Academy of Sciences, vol. **74**, no. 2 (1894), pp. 1–30. Translated from the Russian by J. Shohat.

Velmin, V. P. On the approximate expression of e^x in terms of algebraic functions. Memorial volume dedicated to D. A. Grave [Sbornik posvjaščenii pamjati D. A. Grave], Moscow, 1940, pp. 34–47. (Russian) [MF 3500]

A polynomial $\Phi(x, y)=\varphi_0(x)y^m+\varphi_1(x)y^{m-1}+\cdots+\varphi_m(x)$, where φ_k has degree not greater than n_k ($k=0, \cdots, m$) and $\Phi(x, e^x)=A_\sigma x^\sigma+A_{\sigma+1}x^{\sigma+1}+\cdots$ ($A_\sigma\neq 0$), is said to be of maximal order with respect to $m, n_0, \cdots, n_m$ if σ has its greatest possible value for the given m and n. Such a polynomial is unique up to a constant factor. The number of parameters in the φ_k shows that $\sigma\geqq m+\sum n_k$. The polynomial $\Psi(x, y)=\psi_0(x)y^{m-1}+\cdots+\psi_{m-1}(x)$ is introduced; here $\psi_k(x)$ is the polynomial of the same degree as $\varphi_k(x)$ obtained by setting $\psi_k(x)e^{(m-k)x}$ equal to the (n_m+1)th derivative of $\varphi_k(x)e^{(m-k)x}$. Then $\Psi(x, e^x)=cA_\sigma x^{\sigma-n_m-1}+\cdots$ ($c\neq 0$), and, if σ_p is the maximal σ for $m-p, n_0, \cdots, n_{m-p}$, we have $\sigma_1\geqq\sigma_0-n_m-1$, $\sigma_p\geqq\sigma_{p-1}-n_{m-p+1}-1$, but $\sigma_m=n_0$. Hence $\sigma=m+\sum n_k$, $\sigma_1=\sigma-n_m-1$, and $\Psi(x, y)$ is maximal for $m-1, n_0, \cdots, n_{m-1}$. Conversely, recurrence formulas are found for building up maximal polynomials. The special cases $n_k=k$ ($k=0, \cdots, m$); $m=1$; and $n_0=n_1=\cdots=n_m$ are treated. The case $m=1$, $n_0=n_1$ leads to the Euler-Lambert continued fraction for e^x. Some of the statements lack rigorous proof. *G. Pall* (Princeton, N. J.).

Beale, Frank S. On a certain class of orthogonal polynomials. Ann. Math. Statistics **12**, 97–103 (1941). [MF 4007]

The author studies, with regard to orthogonality, the polynomials $P_n(n, x)$, where

$$P_n(k, x)\equiv D^{n-k}y^{-1}(d^n/dx^n)[D^k y], \quad n\geqq 0;\ k \text{ integral},$$

y being a non-trivial solution of the differential equation

$$\frac{1}{y}\frac{dy}{dx}=\frac{a_0+a_1x}{b_0+b_1x+b_2x^2}\equiv\frac{N}{D}, \qquad a_i, b_i \text{ real}.$$

The study is based on the recurrence relation satisfied by $P_n(n, x)$. *J. A. Shohat* (Philadelphia, Pa.).

Jackson, Dunham. Note on certain orthogonal polynomials. Bull. Amer. Math. Soc. **47**, 96–102 (1941). [MF 3818]

The classical orthogonality property of the "kernel polynomials" $K_n(s, t)$ (namely that they are orthogonal to any polynomial of degree less than n with weight function $\rho(s)(s-t)$, if $\rho(s)$ denotes the original weight function) is extended here to orthogonal systems of trigonometric sums. In order to preserve the non-negative characters of the weight-function, it is assumed that $\rho(s)$ vanishes in a part of $-\pi$, $+\pi$. Then the orthogonality of the kernel polynomials is shown with the weight

$$\sin\tfrac{1}{2}(s-x_1)\sin\tfrac{1}{2}(s-x_2)\rho(s).$$

Similar considerations are possible for polynomials of two real variables orthogonal (with a given weight) on a given algebraic curve. This is illustrated in the example of a curve of third degree. *G. Szegö* (Stanford University, Calif.).

Krall, H. L. On derivatives of orthogonal polynomials. II. Bull. Amer. Math. Soc. **47**, 261–264 (1941). [MF 4171]

It is known that, if the derivatives of a set of orthogonal polynomials are themselves to form an orthogonal set, the original polynomials must satisfy a certain form of linear differential equation of the second order. The same conclusion holds with reference to "generalized orthogonal polynomials" of the kind discussed by the author in previous communications [cf. Pennsylvania State College Studies, no. 6, 24 pp. (1940); these Rev. **2**, 98]. The differential equation belongs to one or another of four types, according to the nature of the quadratic polynomial which forms the coefficient of the second derivative. Three of these types correspond to Jacobi, Laguerre and Hermite polynomials. The fourth type does not correspond to any system of orthogonal polynomials in the ordinary sense, but is shown to yield a set of generalized orthogonal polynomials.

D. Jackson (Minneapolis, Minn.).

Geronimus, J. Sur quelques équations aux différences finies et les systèmes correspondants des polynômes orthogonaux. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 536–538 (1940). [MF 3966]

The author studies the properties of the solution y_n of the difference equation

$$\text{(I)} \qquad y_n=(z-\alpha_n)y_{n-1}-\lambda_n y_{n-2}, \qquad \lambda_n\neq 0,$$

under various conditions imposed on the constants α_n, λ_n (existence of limiting relations, etc.). Particular attention is paid to the case of real bounded sequences $\{\alpha_n\}$, $\{\lambda_n\}$ such that $\lambda_{s+k}=\lambda_r=l_r$, $\alpha_{r+k}=\alpha_r=a_r$ ($r\geqq s+1$), $\lambda_n>0$ ($n=1, 2, \cdots$). It is shown that here (I) has for solutions a sequence of orthogonal polynomials on a finite interval. The interval of orthogonality and the weight-function are characterized explicitly. In the special case $k=1$, $\lambda_{s+1}=\lambda_{s+2}=\cdots=\frac{1}{4}$, $\alpha_{s+1}=\alpha_{s+2}=\cdots=0$, the author also gives an asymptotic expression for the said polynomials of very high degree. All this is closely related to the algebraic function

$$\phi(z)=([P_k^{(1)}(z)-l_1 P_{k-2}^{(2)}(z)]^2-4l_1l_2\cdots l_k)^{\frac{1}{2}}.$$

J. A. Shohat (Philadelphia, Pa.).

Gillis, J. Tchebycheff polynomials and the transfinite diameter. Amer. J. Math. **63**, 283–290 (1941). [MF 4150]

By the use of Tchebycheff polynomials the following theorem of Fekete is proved, which was communicated to the author without proof: Let $E=\sum E_r$, E_r closed, E in a circle with diameter 1; then

$$f\{\tau(E)\}\leqq\sum f\{\tau(E_r)\}, \quad f(x)=\{-\log x\}^{-1},$$

where $\tau(E)$ is the transfinite diameter of E. Further it is shown that this theorem is the best possible in the sense that $f(x)$ can not be replaced by another function $g(x)$ for which $g(x)/f(x)\to 0$ as $x\to 0$. Finally the previous results are extended to a general metric space. *G. Szegö.*

Koschmieder, Lothar. Summierung einer nach den Hermiteschen Polynomen des Kreises fortschreitenden Reihe. Anz. Akad. Wiss. Wien **1940**, 41–43 (1940). [MF 3620]

Proof of the formula

$$\sum_{n=0}^{\infty}s^n\sum_{l+m=n}\frac{l!m!}{(l+m)!}U_{lm}(x, y)U_{lm}(\xi, \eta)$$
$$=\{[1-2s(x\xi+y\eta)+s^2]^2-4s^2(1-x^2-y^2)(1-\xi^2-\eta^2)\}^{-1},$$

where $x^2+y^2<1$, $\xi^2+\eta^2<1$, and

$$[(1-sx-ty)^2+(s^2+t^2)(1-x^2-y^2)]^{-\frac{1}{2}}=\sum s^l t^m U_{lm}(x, y),$$
$$(1-2sx-2sy+s^2+t^2)^{-1}=\sum s^l t^m V_{lm}(x, y).$$

E. Hille (New Haven, Conn.).

Varma, R. S. An infinite series involving the product of Bessel functions and generalised Laguerre polynomials. Proc. Indian Acad. Sci., Sect. A. **12**, 532–534 (1940). [MF 3936]

By means of operational methods an expression for

$$\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha+n+1)}x^{n/2}J_n(2x^{\frac{1}{2}})L_n^{(\alpha)}(y)$$

is found in terms of ${}_0F_2$. *G. Szegö.*

Cowling, T. G. On certain expansions involving products of Legendre functions. Quart. J. Math., Oxford Ser. **11**, 222–224 (1940). [MF 3131]

Two formulas are proved representing

$$(\mu^2+\mu'^2-1)^{n/2}P_n^m(\alpha),\quad (\mu^2+\mu'^2-1)^{-\frac{1}{2}(n+1)}P_n^m(\alpha),$$
$$\alpha=\mu\mu'(\mu^2+\mu'^2-1)^{-\frac{1}{2}},$$

in terms of $P_r^m(\mu)P_r^m(\mu')$ and $P_r^m(\mu)Q_r^m(\mu')$, respectively. Here $0\leqq m\leqq n$; in the first representation $0\leqq r\leqq n$, in the second one $n\leqq r\leqq\infty$. *G. Szegö.*

Feldheim, E. Une propriété caractéristique des polynômes de Laguerre. Comment. Math. Helv. **13**, 6–10 (1940). [MF 3985]

The author shows that the only polynomials $\Phi_n(x)$ of degree n satisfying a multiplication relation of the form

$$\lambda^n\Phi_n(x/\lambda)=\sum_{k=0}^{n}A_{nk}(\lambda-1)^{n-k}\Phi_k(x)$$

are given by

$$\Phi_n(x)=b_n\sum_{k=0}^{n}c_k x^k/(n-k)!\qquad\text{with } A_{nk}=b_n/b_k(n-k)!.$$

If, in particular, the $\Phi_n(x)$ form an orthogonal system, then

$$\frac{1}{b_n}\Phi_n(x)=\frac{c_0\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)}L_n^{(\alpha)}\left(-\frac{x}{\rho}\right),$$

where $\Re(\alpha)>-1$, ρ is arbitrary and $L_n^{(\alpha)}(u)$ is the Laguerre polynomial. *E. Hille* (New Haven, Conn.).

Erdös, P. On divergence properties of the Lagrange interpolation parabolas. Ann. of Math. (2) **42**, 309–315 (1941). [MF 3686]

Denote by $L_nf(x)$ the Lagrange interpolation parabola agreeing with the given function $f(x)$ $(-1\leqq x\leqq 1)$ in the zeros of the nth Tchebycheff polynomial $T_n(x)$. In connection with the convergence of $L_nf(x)$ to $f(x)$ the exceptional role of the following set S of points is now demonstrated, where S is the set of abscissae of the form

$$S: x=\cos(p\pi/q),\quad 0<p<q;\ p \text{ and } q \text{ both odd};\ (p, q)=1.$$

The two theorems proved are as follows: (I) If $x_0\varepsilon S$, then there exists a continuous $f(x)$ such that $L_nf(x_0)\to+\infty$. There are no such $f(x)$ if $x_0\bar{\varepsilon}S$, as stated by the following theorem: (II) If $-1<x_0<1$ and x_0 does not belong to S, then for every continuous $f(x)$ there is a suitable sequence $n_1<n_2<\cdots$ such that $L_{n_i}f(x_0)\to f(x_0)$. The fact that the author's Lemma 1 requires some additional condition like $(m, n)=1$ (which is not there stated) seems to necessitate additional arguments in two points of the proof of Theorem 1. *I. J. Schoenberg* (Waterville, Me.).

Grünwald, G. On a convergence theorem for the Lagrange interpolation polynomials. Bull. Amer. Math. Soc. **47**, 271–275 (1941). [MF 4174]

Denote by

$$L_n(f; x)=\sum_{k=1}^{n}fx_k^{(n)}l_k^{(n)}(x)$$

the Lagrange interpolation polynomial of $f(x)$ taken at the roots of $T_n(x)$. The author proves the following theorem: Let $f(x)$ be continuous in $(-1, +1)$; then

$$\lim_{n\to\infty}\tfrac{1}{2}[L_n(f; \theta-\pi/2n)+L_n(f; \theta+\pi/2n)]=f(x),\quad x=\cos\theta,$$

uniformly in $(-1, +1)$. Recently Kharchiladzé proved a similar and more general result [Rec. Math. [Mat. Sbornik] N.S. **8** (**50**), 471–487 (1940), in particular p. 486; cf. these Rev. **2**, 196]. *P. Erdös* (Philadelphia, Pa.).

Grünwald, G. Note on interpolation. Bull. Amer. Math. Soc. **47**, 257–260 (1941). [MF 4170]

Let $x_1^{(1)}$; $x_1^{(2)}<x_2^{(2)}$; $\cdots$; $x_1^{(n)}<x_2^{(n)}<\cdots<x_n^{(n)}$; $\cdots$ be given, $-1<x_i^{(n)}<1$. Let $f(x)$ be any continuous function, and put

$$I(x; f)=\sum_{k=1}^{n}f(x_k^{(n)})q_k^{(n)}(x),$$

where $q_k^{(n)}(x_k^{(n)})=1$ and $q_k^{(n)}(x_i^{(n)})=0$, $i\neq k$. The question arises whether it is possible to determine, for a given sequence $x_i^{(n)}$, the fundamental polynomials $q_1^{(1)}(x)$; $q_1^{(2)}(x)$, $q_2^{(2)}(x)$; $\cdots$; $q_1^{(n)}(x)$, $q_2^{(n)}(x)$, $\cdots$, $q_n^{(n)}(x)$; $\cdots$ so that for an arbitrary continuous function $f(x)$ $\lim I(x; f)=f(x)$ uniformly in $(-1, +1)$. We clearly must suppose that the sequence $x_i^{(n)}$ is everywhere dense in $(-1, +1)$, that is, $\max_i(x_{i+1}^{(n)}-x_i^{(n)})\to 0$ when $n\to\infty$. The author proves that this trivial necessary condition is also sufficient.

P. Erdös (Philadelphia, Pa.).

Kamenetzky, I. M. Sur l'interpolation au moyen des dérivées et les procédés d'interpolation correspondants. II. C. R. (Doklady) Acad. Sci. URSS (N.S.) **26**, 217–219 (1940). [MF 3553]

[The first part appeared in the same C.R. **25**, 356–358 (1939); cf. these Rev. **1**, 334.] The interpolation problem $f^{(n_i)}(x_i)=c_i$ $(i=0, 1, \cdots)$ arising from a graded sequence $n_0\leqq n_1\leqq n_2\leqq\cdots$ [see these Rev. **1**, 334] is formally solved by a development (1) $f(x)=\sum_{i=0}^{\infty}f^{(n_i)}(x_i)p_i(x)$, the basic set of polynomials $p_i(x)$ depending on the sequences (n_i) and (x_i). The graded sequence is now said to be periodic, of period m, if (2) $u_{i+m}=u_i+m$ and (3) $x_{i+m}=x_i$ $(i=0, 1, \cdots)$. For this case the author determines the best constant l (as the least positive absolute value of the roots of a certain transcendental equation) such that entire functions of class less than $[1, l]$ may be represented by the expansion (1). The special cases $n_i=i$ and $n_i=m[i/m]$ were treated by Gontcharoff and Poritsky, respectively. These results are also extended to a case of asymptotic periodicity, that is, (2) is retained, but (3) replaced by $\lim_{i\to\infty}x_{i+km}=\alpha_i$ $(i=0, 1, \cdots, m-1)$. The last theorem announced in this paper (extending a result by the reviewer) is as follows: If x_i is a complex sequence such that all its limit points (as a set, not as a sequence) lie in the interval $[-1, 1]$ then any entire function $f(x)$, of class less than $[1, \pi/4]$, is represented in the whole plane by the generalized Abel series associated with the sequence x_i. *I. J. Schoenberg* (Waterville, Me.).

Lozinski, S. Sur la forte convergence des procédés d'interpolation. C. R. (Doklady) Acad. Sci. URSS (N.S.) 30, 386–390 (1941). [MF 4376]

The author states without proof a number of results concerning (a) approximation of absolutely continuous functions by the partial sums of their Fourier series; (b) approximation of absolutely continuous functions $F(x)$ by the Lagrange interpolating trigonometric polynomials $U_n(F, x)$ and by the arithmetic means $V_n(F, x)$ of the first $n+1$ partial sums of $U_n(F, x)$ (Bernstein's polynomials); (c) the approximation in the mean of integrable functions $f(x) = F'(x)$ by the derived polynomials $U_n'(F, x)$ and $V_n'(F, x)$.

A. Zygmund (South Hadley, Mass.).

Lozinski, S. Über singuläre Integrale. Rec. Math. [Mat. Sbornik] N.S. 7 (49), 329–363 (1940). (German. Russian summary) [MF 2796]

Let $\phi_n(t, x)$ be measurable and essentially bounded in the square $a \leqq t \leqq b$, $a \leqq x \leqq b$, and $f(x)$ Lebesgue integrable over (a, b). The author considers the "singular integral" $f_n(x) = \int_a^b \phi_n(t, x) f(t)\,dt$ and its convergence to f in the space L of integrable functions. He also discusses the problem of $(\mathfrak{A}, \mathfrak{B})$ convergence of $f_n(x)$ which means the necessary and sufficient conditions in order that relation $\int_a^b f_n(x) g(x)\,dx \to \int_a^b f(x) g(x)\,dx$ should hold for any pair of functions f, g, where $f \in \mathfrak{A}$, $g \in \mathfrak{B}$. In addition to the class L the author also considers the class M of measurable and essentially bounded functions, and the class V of functions of bounded variation, with the usual definition of norms. The following results are established. In order that f_n converge to f in L it is necessary and sufficient that this would be so for all f out of a set F dense in L, and also that

$$(1) \qquad \operatorname*{ess\,sup}_{a \leq t \leq b} \int_a^b |\phi_n(t, x)|\,dx \leqq K, \qquad n = 1, 2, \cdots.$$

[This theorem was proved by another method by Kantorovitch and Vulich, Compositio Math. 5, 119–165 (1937), and is a generalization of a well-known theorem of Hahn, Denkschriften Akad. Wiss. Wien 93, 565–693 (1917).] In order that f_n would converge (L, M) it is necessary and sufficient that it would converge (F, M) and also (1) be satisfied, while for the convergence (M, L) it is necessary and sufficient that f_n would converge (M, F) and the condition

$$(2) \qquad \operatorname*{ess\,sup}_{a \leq x \leq b} \int_a^b |\phi_n(t, x)|\,dt \leqq K, \qquad n = 1, 2, \cdots,$$

be satisfied. In order that f_n would converge (L, V) it is necessary and sufficient that f_n converge in (F, V) and the condition

$$(3) \qquad \operatorname*{ess\,sup}_{a \leq t \leq b} \left| \int_a^b \phi_n(t, x)\,dx \right| \leqq K, \qquad n = 1, 2, \cdots,$$

be satisfied, while for the convergence (V, L) it is necessary and sufficient that f_n converge (V, F) and the condition

$$(4) \qquad \operatorname*{ess\,sup}_{a \leq x \leq b} \left| \int_a^b \phi_n(t, x)\,dt \right| \leqq K, \qquad n = 1, 2, \cdots,$$

be satisfied. The author applies these results to the question of convergence of orthogonal expansions and also of some linear interpolation processes. *J. D. Tamarkin.*

Obreschkoff, N. Neue Quadraturformeln. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. 1940, no. 4, 20 pp. (1940). [MF 3999]

Assume $f(x)$ possesses in a given interval (a, b) $n+k+1$ derivatives. The author derives a formula for the kth Cesàro sum of the first $n+1$ terms of the Taylor expansion of $f(x)$ in (a, b). Its remainder is given in the form of a definite integral. Letting $f'(x) = F(x)$, this yields a quadrature formula for $\int_a^b F(x)\,dx$. By specifying n and k, various formulae are obtained, some generalizing known ones, like the trapezoidal rule, etc. It is shown, by actual computation, that some of these new formulae are capable of giving better numerical results, and with less labor, than Simpson's rule or the Euler-Maclaurin formula. *J. A. Shohat.*

Tietz, A. Angenäherte Berechnung n-facher Integrale. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 423–464 (1940). (Russian. German summary) [MF 4084]

The author reduces, by a suitable change of variables, the n-uple integral under consideration to

$$(1) \qquad {}_0J_1 = \int_0^1 \int_0^1 \cdots \int_0^1 \omega(y_1, y_2, \cdots, y_n)\,dx_1 dx_2 \cdots dx_n, \qquad y_k = x_k^2.$$

He now seeks an approximate evaluation of (1) by means of the expression

$$(2) \qquad \sum_i \lambda_i \omega(\xi x_1^2, \xi x_2^2, \cdots, \xi x_n^2), \quad 0 < \xi x_k^2 \leqq 1; \; k = 1, 2, \cdots, n,$$

first assuming ω in (1) to be a polynomial in the x's. Methods are developed for the choice and the computation of the abscissas ξx_k and the coefficients λ_i in (2) so as to insure greater accuracy or greater simplicity of computation; also (2) is supplemented by terms involving the partial derivatives of ω. Finally, (1) is compared to the simple integral $\int_0^1 \omega(y, y, \cdots, y)\,dx$, $y = x^2$. Thus various quadrature formulae are derived, extending, among other things, to the n-dimensional space the classical formulae of Cotes, Gauss, Tchebycheff. (2) is then extended to any ω admitting a Maclaurin expansion. The latter furnishes the remainder term for each quadrature formula considered. The author gives tables with numerical values of the ξx_k and λ_i. He closes by giving several concrete numerical applications. [In the bibliography these should have been added a paper by Appell, Ann. Fac. Sci. Univ. Toulouse (1890), one of the first, if not the first, on this subject.] *J. A. Shohat.*

Special Functions

Chowla, S. and Auluck, F. C. Some properties of a function considered by Ramanujan. J. Indian Math. Soc. (N.S.) 4, 169–173 (1940). [MF 4322]

Consider the function

$$\phi(t) = \sum_{n=1}^{\infty} (n^{n-2}/(n-1)!)\, t^{n-1} e^{-nt}.$$

The authors prove that $\operatorname{sgn} \phi^{(r)}(t) = (-1)^r$ for $r = 1, 2, 3$, and conjecture that this holds for all r. Put $e^{\omega}/\omega = e^t/t$, $t \geqq 1$, $0 < \omega \leqq 1$, and expand $\omega - 1$ in powers of $(1-t)$ as $\sum_{n=1}^{\infty} c_n (1-t)^n$. The authors conjecture that $nc_n \to 1$.

P. Erdös (Philadelphia, Pa.).

Toscano, Letterio. Sul prodotto di due polinomi di Laguerre e di Hermite. Atti Accad. Italia. Rend. Cl. Sci. Fis. Mat. Nat. 1, 405–411 (1940). [MF 3618]

Se rattachant à des résultats récemment publiés sur les

polynômes de Laguerre, l'auteur établit la formule

$$L_{m+n}^{(\alpha)}(x)=\frac{m!n!}{(m+n)!}\Gamma(1+\alpha+m+n)$$

$$\times\sum_{i=0}^{n}\frac{(-1)^i\Gamma(\alpha+i)\Gamma(1+\alpha+2i)x^{2i}L_{m-i}^{(\alpha+2i)}(x)L_{n-i}^{(\alpha+2i)}(x)}{i!\Gamma(\alpha+2i)\Gamma(1+\alpha+m+i)\Gamma(1+\alpha+n+i)},\quad m\geqq n.$$

À l'aide d'une relation limite [Boll. Un. Mat. Ital. 1, 337–339 (1939); ces Rev. 1, 116], il déduit ensuite des formules concernant les polynômes de Laguerre des relations de même nature sur les polynômes d'Hermite. Telles relations sont, p.e., celles établies par le rapporteur [J. London Math. Soc. 13, 22–29 (1938); Nederl. Akad. Wetensch., Proc. 43, 224–248, 379–386 (1940); ces Rev. 1, 232] par des méthodes toutes différentes. L'auteur fait observer que la formule concernant le développement du produit de deux polynômes d'Hermite, un résultat valable aussi pour les fonctions du cylindre parabolique, ainsi que son inversion, sont déjà considérées par N. Nielsen [Danske Vid. Selsk. I. 6, 33 (1918)].

E. Feldheim (Budapest).

Erdélyi, A. On Lamé functions. Philos. Mag. (7) 31, 123–130 (1941). [MF 3885]

Lamé's equation

$$y''+[h-n(n+1)k^2s^2]y=0,\quad s=\operatorname{sn}(u,k),$$

has for certain values of n and $h=h(n)$ a solution with a real and at the same time another with an imaginary period. They are only simply-periodic, and coincide only in case of Lamé polynomials which are the only double-periodic solutions of the equation mentioned. For Lamé polynomials associated with complementary moduli a transformation formula is derived. *G. Szegö.*

Brainerd, J. G. and Weygandt, C. N. Solutions of Mathieu's equation. I. Philos. Mag. (7) 30, 458–477 (1940). [MF 3709]

In this paper the authors are concerned with non-periodic solutions of Mathieu's equation

$$y''+\epsilon(1+k\cos t)y=0. \tag{1}$$

They point out that the Mathieu functions are periodic solutions of this equation which exist only for certain pairs of values of parameters ϵ and k, whereas many practical problems require solutions where the values of ϵ and k are prescribed by the physical conditions.

Assuming that ϵ and k have fixed values the authors discuss two primary solutions $g(t)$ and $h(t)$ defined by the initial conditions $g(0)=1$, $g'(0)=0$; $h(0)=0$, $h'(0)=1$. Two auxiliary periodic functions u_1 and u_2 are also introduced:

$$g(t)=u_1\cos\mu t-u_2\sin\mu t,$$

$$H(t)=(1-b^2)^{\frac{1}{2}}h(t)/h(2\pi)=u_1\sin\mu t+u_2\cos\mu t,$$

where b is the value of g or h' at $t=2\pi$, and the characteristic exponent μ is found from the relation

$$b+(b^2-1)^{\frac{1}{2}}=e^{i2\pi\mu}.$$

Various properties of these functions are developed and it is shown, for example, that the solution of (1) is completely determined when g and h are known in an interval π of t, or when either g or h is known in an interval 2π of t. The paper also includes sample curves of the g and h solutions, and a table of values of the parameter b for assigned values of ϵ and k within the range $0\leqq\epsilon\leqq10$, $0\leqq k\leqq1$.

M. C. Gray (New York, N. Y.).

Bickley, W. G. Notes on Mathieu functions. I. A class of hyperbolic Mathieu functions. Philos. Mag. (7) 30, 312–322 (1940). [MF 3183]

The characteristic numbers a_n, b_n and the corresponding periodic solution $\mathrm{ce}_n(\eta,\theta)$, $\mathrm{se}_n(\eta,\theta)$ of the Mathieu equation in the form

$$\frac{d^2y}{d\eta^2}+(a-2\theta\cos2\eta)y=0$$

used by Ince are adopted to obtain analogous solutions $\mathrm{Ce}_n(\xi,\theta)$, $\cdots$, $\mathrm{Cei}\,c_n(\xi,\theta)$ of the related equations

$$\frac{d^2y}{d\xi^2}-(a-2\theta\cosh2\xi)y=0,\qquad \frac{d^2y}{d\xi^2}-(a+2\theta\cosh2\xi)y=0.$$

Expansions in hyperbolic and Bessel functions of ξ are obtained. Integral formulas are derived for them and their asymptotic behavior for large ξ is investigated. Some of the above functions are of interest in the problems of the Oseen approximations to the flow of a viscous fluid around a cylinder and the problem of cooling of a cylinder by forced convection. *H. Poritsky* (Schenectady, N. Y.).

Copson, E. T. On an infinite integral connected with the theory of Bessel functions. Proc. Cambridge Philos. Soc. 37, 102–104 (1941). [MF 3769]

A series is found for the integral $\int_0^\infty \sin(xu)(1+u^2)^{-s-1}du$ when $x>0$ and $\Re(s+1)>0$, and when $s\to0$ it is found that the limiting value of the power series is

$$-(\log x)(\operatorname{sh}x)+\sum_0^\infty x^{2m+1}\psi(2m+2)/(2m+1)!.$$

An alternative method, due to G. H. Hardy, depends on some properties of the integral logarithm. *H. Bateman.*

Wright, E. M. The generalized Bessel function of order greater than one. Quart. J. Math., Oxford Ser. 11, 36–48 (1940). [MF 1865]

The function considered by the author is the integral function

$$\Phi(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+1)\Gamma(\rho n+\beta)}.$$

In an earlier paper [Proc. London Math. Soc. (2) 38, 257–270 (1934)] asymptotic expansions of $\Phi(z)$ were developed for large z and real positive values of ρ. In the present paper the appropriate asymptotic expansions are obtained in the strip $-1<\rho<0$.

Introducing the notation $\sigma=-\rho$, $0<\sigma<1$; $y=-z$, $-\pi<\arg z$, $y\leqq\pi$; $Y(y)=(1-\sigma)[\sigma^\sigma y]^{1/(1-\sigma)}$; $Y_1=Y(ze^{i\pi})$, $Y_2=Y(ze^{-i\pi})$, the author defines an "exponential" asymptotic expansion:

$$I(Y)=Y^{1-\beta}e^{-Y}\Big(\sum_{m=0}^{M-1}A_mY^{-m}+O(Y^{-M})\Big),$$

and an "algebraic" asymptotic expansion:

$$J(z)=\sum_{l=0}^{L-1}\frac{z^{(\beta-1-l)/\sigma}}{\sigma\Gamma(l+1)\Gamma[1+(\beta-l-1)/\sigma]}+O[z^{(\beta-1-L)/\sigma}].$$

Then the following theorems are proved: (1) If $|\arg y|\leqq\min(\frac{1}{2}\pi(1-\sigma),\pi)-\epsilon$, then $\Phi(z)=I(Y)$. (2) If $0<\sigma<\frac{1}{2}$ and $|\arg z|\leqq\pi(1-\sigma)-\epsilon$, then $\Phi(z)=I(Y_1)+I(Y_2)$. (3) If $\sigma=\frac{1}{2}$ and $|\arg z|\leqq\pi(1-\sigma)-\epsilon$, then $\Phi(z)=I(Y_1)+I(Y_2)+J(z)$. (4) If $\frac{1}{2}<\sigma<1$ and $|\arg z|\leqq\frac{1}{2}\pi(3\sigma-1)-\epsilon$, then $\Phi(z)=J(z)$. (5) If $\frac{1}{2}<\sigma<1$ and $|\arg z\pm\frac{1}{2}\pi(3\sigma-1)|\leqq\pi(1-\sigma)$

$-\epsilon$, then $\Phi(z)=I(Y)+J(z)$. (6) If $\sigma=\frac{1}{2}$ and 2β is an integer the $J(z)$ of theorem 5 is a finite polynomial. *M. C. Gray.*

Wright, E. M. The asymptotic expansion of the generalized hypergeometric function. Proc. London Math. Soc. (2) **46**, 389–408 (1940). [MF 2823]

In an earlier paper [J. London Math. Soc. **10**, 287–293 (1935)] the author published, without proof, six theorems on the asymptotic expansion of the generalized hypergeometric function for all large values of the argument. Proofs of the first five theorems were later included in another paper [Philos. Trans. Royal Soc. Ser. A. **238**, 423–451 (1940); these Rev. **1**, 212], and the present paper supplies the proof of the sixth theorem.

The function considered is

$$ {}_pF_q(z)=\sum_{n=0}^{\infty}\frac{f(n)}{\Gamma(n+1)}z^n, $$

where

$$ f(t)=\prod_{r=1}^{p}\Gamma(\beta_r+\alpha_r t)\left[\prod_{r=1}^{q}\Gamma(\mu_r+\rho_r t)\right]^{-1} $$

and is regular for integral values of t. The numbers α_r, ρ_r are all real and positive and

$$ \kappa=1+\rho_1+\cdots+\rho_q-\alpha_1-\cdots-\alpha_p>0. $$

Writing $\zeta=\arg z$, $\eta=\arg(-z)$, $-\pi<\zeta$, $\eta\leqq\pi$, and

$$ h=\prod_{r=1}^{p}\alpha_r^{\alpha_r}\prod_{r=1}^{q}\rho_r^{-\rho_r}, $$

the author introduces new variables

$$ Z=\kappa(h|z|)^{1/\kappa}e^{i\zeta/\kappa},\quad Z_1=|Z|e^{i(\zeta+\pi)/\kappa},\quad Z_2=|Z|e^{i(\eta-\pi)/\kappa}, $$

and the theorem proved in the paper may then be written: If $f(t)$ has only a finite number of poles or none, then $\kappa\geqq 1$, and the asymptotic expansion of ${}_pF_q$ is given by

$$ {}_pF_q(z)=I(Z_1)+I(Z_2)+H(-z),\qquad 1<\kappa<2, $$

$$ {}_pF_q(z)=I(Z)+H(-z),\qquad \kappa=1, $$

where $I(X)$ is an "exponential" asymptotic expansion

$$ I(X)=X^{\delta}e^{X}\left[\sum_{m=0}^{M-1}A_mX^{-M}+O(X^{-M})\right], $$

and $H(y)$ is a finite algebraic expansion

$$ H(y)=\sum_{r=0}^{p}\sum_{l=0}^{l_r}P_{r,l}y^{-(l+\beta_r)/\alpha_r}. $$

M. C. Gray (New York, N. Y.).

Shabde, N. G. On some results involving confluent hypergeometric functions. J. Indian Math. Soc. (N.S.) **4**, 151–157 (1940). [MF 4319]

With the aid of the Laplace transformation expansions are found for the integrals

$$ \int_0^x k_{2n}(x-t)t^{-\frac{1}{2}}dt,\quad \int_0^x L_n(2x-2t)t^{-\frac{1}{2}}dt, $$

$$ \int_0^x (x-t)^{-\frac{1}{2}}L_n^{\alpha}(2t)e^{-\frac{1}{2}t}t^{\alpha}dt $$

in series involving squares of the parabolic cylinder function $D_{-n}^2(i\sqrt{2}x)$. Some infinite integrals involving Bessel functions and confluent hypergeometric functions are expressed by means of series and a few operational representations of products of Bessel functions and confluent hypergeometric functions are given. *H. Bateman* (Pasadena, Calif.).

Badarau, Gabriel. Sur l'expression du coefficient de transmission d'une barrière de potentiel coulombien en termes de fonctions développables en série. Acad. Roum. Bull. Sect. Sci. **22**, 396–400 (1940). [MF 3201]

The logarithmic case of the confluent hypergeometric equation $zy''+(b-z)y'-ay=0$, which has already been discussed by W. J. Archibald [Philos. Mag. (7) **26**, 415–419 (1938)] for a positive integral value of b, is discussed here with the aid of Whittaker's equation $4z^2M''+M(1-4m^2+4kz-z^2)=0$, m being half an odd integer. The general solution is compared with the normalized solutions of Yost, Wheeler and Breit and so an asymptotic expansion is obtained. The results are used to calculate the coefficient of transmission of a Coulombian potential barrier chosen so that $U=U_i=$constant for $0\leqq r<r_0$ and $U=2Ze^2/r$ for $r>r_0$. For the region $r<r_0$, use is made of a Legendre series in which the coefficients involve Bessel function of half an odd integer. *H. Bateman* (Pasadena, Calif.).

Toscano, Letterio. Numeri di Stirling generalizzati, operatori differenziali e polinomi ipergeometrici. Pont. Acad. Sci. Comment. **3**, 721–757 (1939). [MF 4107]

This paper contains 235 numbered formulas relating to the topics mentioned in the title. In earlier papers [see e.g., Atti Ist. Veneto Sci. **96** (1936–1937)] the author has considered associated operations X and A, satisfying the commutator relation $AX-XA=I$. Specializing X to be differentiation with respect to a power of x, he obtains a large number of relations between different differential operators in which various types of Stirling coefficients figure. These relations applied to Jacobi polynomials, which themselves are derivatives, give rise to numerous relations between such polynomials which by suitable specialization and passage to the limit give corresponding relations between Hermite and Laguerre polynomials, etc. *E. Hille.*

Pevnyi, B. G. Some functional equations for generalized hypergeometric series and Whittaker's functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 310–312 (1940). [MF 3333]

A functional equation of the following form is given without proof:

$$ {}_pF_q\left(\begin{matrix}a_1, a_2, \cdots, a_p\\ b_1, b_2, \cdots, b_q\end{matrix}; z\right)=\lambda\int(-t)^{b_1'-1}(t-1)^{b_1-b_1'-1}\cdot {}_pF_q\left(\begin{matrix}a_1, a_2, \cdots, a_p\\ b_1', b_2, \cdots, b_q\end{matrix}; tz\right)dt, $$

where the integration is extended over a closed contour of the type $(1+, 0+, 1-, 0-)$ and λ is a proper constant. Various special cases and applications are indicated.

G. Szegö (Stanford University, Calif.).

Pevnyi, B. G. On the asymptotic expansion of Whittaker's functions. C. R. (Doklady) Acad. Sci. URSS (N.S.) **28**, 308–309 (1940). [MF 3332]

The interesting results of this note are stated without proof. They deal with the remainder $R_n(z)$ of the asymptotic expansion

$$ e^{-z/2}z^kM_{k,m}(z)\sim\sum_{\nu=0}^{\infty}c_\nu z^{-\nu},\quad R_n(z)=\sum_{\nu=0}^{n}c_\nu z^\nu $$

of Whittaker's function $M_{k,m}(z)$, $|\arg z|<\pi/2$, $z\to\infty$. Let

m and k be real, $n>|m|-k-\frac{1}{2}$; then $R_n(z)=0$ has a root for which an asymptotic formula is obtained as $n\to\infty$. A similar formula for n as $z\to\infty$ is also given. *G. Szegö.*

Meijer, C. S. Neue Integraldarstellungen für Whittakersche Funktionen. I. Nederl. Akad. Wetensch., Proc. **44**, 81–92 (1941). [MF 4048]

An addition theorem is given for a function of four positive integers m, n, p, q and of a variable w and two sets of parameters $a_1, a_2, \cdots, a_p; b_1, b_2, \cdots, b_q$. The function, in which the integers are replaced by $n+\mu, m+\nu, q+\sigma, p+\tau$, respectively, is expressed as an integral of the product of two of the functions, the first having the integers m, n, p, q, while the second has the integers μ, ν, σ, τ. The variable is in each case a multiple of the variable of integration x, while the parameters are arbitrary except for certain restrictions expressed by the requirements that certain combinations of the parameters must not be integers and by certain inequalities. The function is expressed first as a series involving generalized hypergeometric functions and gamma functions. An equivalent representation is then found as an integral with respect to s of w^s and the ratio of two sets of products of gamma functions as additional factor. The path runs from $-i\infty+\lambda$ to $i\infty+\lambda$ so that certain points are on its left and certain other points on its right. Several cases are distinguished and considered separately in the proof. *H. Bateman* (Pasadena, Calif.).

Erdélyi, A. Integral representations for Whittaker functions. Proc. Benares Math. Soc. **1**, 39–53 (1939). [MF 1530]

Let $z\neq 0$, $\frac{1}{2}>\Re k+|\Re m|$, $2m$ not an integer. The expression

$$\Gamma(\tfrac{1}{2}-k-m)\int_0^\infty e^{-t}t^{-\frac{1}{2}-k+m}(1+t/z)^{-\frac{1}{2}+k+m}\,dt$$

remains unchanged if we replace m by $-m$. This fact plays a similar role in the theory of Whittaker's function as certain integral identities do for hypergeometric integrals. The present paper gives various proofs of the fact mentioned proceeding on the same line as the proofs given previously for the hypergeometric case by Riemann, Wirtinger, Poole and the author. In case $2m$ is an integer the formula needs a slight modification. Further integral representations for Whittaker's function are also discussed. *G. Szegö.*

Erdélyi, A. Some confluent hypergeometric functions of two variables. Proc. Roy. Soc. Edinburgh **60**, 344–361 (1940). [MF 3362]

In a former paper [same Proc. **59**, 224–241 (1939); these Rev. **1**, 55] the author dealt with the system of partial differential equations

$$(*)\qquad \begin{aligned} x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial y\partial x}+(\gamma-x)\frac{\partial z}{\partial x}-\beta z=0,\\ x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}+(\gamma-y)-\beta' z=0.\end{aligned}$$

In continuation, the author gives in the present paper first a number of convergent expansions of certain four solutions of (*) obtained in his first paper. Moreover, in correcting the erroneous remark of the first paper that these four solutions are the only ones expressible by convergent hypergeometric series of two variables, he obtains three new solutions, together with convergent expansions. For all solutions thus obtained the author outlines the transformation theory and discusses the monodromic group of (*). Finally he compares his results with those obtained by J. Horn [Monatsh. Math. Phys. **47**, 186–194 (1938) and **47**, 359–379 (1939); these Rev. **1**, 17], and discusses also the equivalence of (*) with certain other known systems of partial differential equations. *E. Rothe* (Oskaloosa, Iowa).

Burchnall, J. L. and Chaundy, T. W. Expansions of Appell's double hypergeometric functions. Quart. J. Math., Oxford Ser. **11**, 249–270 (1940). [MF 3787]

By means of an elegant symbolic notation Appell's series of various types are expressed in terms of hypergeometric functions. As a sample we mention the following formula:

$$F^{(2)}[a;b,b';c,c';x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{m!n!(c)_{m+n}}x^m y^n$$

$$=\frac{\Gamma(a)\Gamma(\delta+\delta'+a)}{\Gamma(\delta+a)\Gamma(\delta'+a)}F(a,b;c;x)F(a,b';c';y),$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)$, $\delta=x\,\partial/\partial x$, $\delta'=y\,\partial/\partial y$. It is impossible to give a more detailed account of the numerous other identities of related nature dealt with in this paper. Conversely, $F(a,b;c;x)F(a,b';c';y)$ can be represented in similar manner in terms of $F^{(2)}$. The rest of the paper is devoted to integral representations and to the discussion of the convergence of the previous expansions. *G. Szegö.*

Differential Equations

Court, Louis M. A theorem on maxima and minima with an application to differential equations. J. Math. Phys. Mass. Inst. Tech. **20**, 99–106 (1941). [MF 3845]

This paper is chiefly concerned with an elementary theorem on the extrema of point functions subject to a special type of linear side-condition. *W. T. Reid* (Chicago, Ill.).

Levinson, N. The growth of the solutions of a differential equation. Duke Math. J. **8**, 1–10 (1941). [MF 3943]

Discussing the differential equation

$$d^2x/dt^2+\phi(t)x=0$$

with $\phi(t)$ real on $0<t<\infty$, the author establishes for a solution $x(t)$ the appraisals

$$(1)\qquad x(t)=0(\exp[\tfrac{1}{2}a^{-1}\alpha(t)]),\qquad \text{as } t\to\infty,$$

with a a positive constant and $\alpha(t)=\int_0^t|\phi(t)-a|\,dt$, and, if $\alpha(t)=0(t)$ as $t\to\infty$,

$$(2)\qquad \limsup_{t\to\infty}|x(t)|\exp[\tfrac{1}{2}a^{-1}\alpha(t)]>0.$$

He shows that each appraisal fails of being a best possible one by at most a constant in the exponent of e. The result refines and extends the previously known facts that $x(t)$ is bounded as $t\to\infty$ if $\alpha(\infty)<\infty$, and not necessarily so if $\phi(t)=a+O(1/t)$. *R. E. Langer* (Madison, Wis.).

Ivanov, V. S. Justification of an hypothesis of van der Pol in the theory of autovibrations. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser. **10**, 111–119 (1940). (Russian) [MF 3306]

The author gives a new proof of the theorem: If $F(x)$ is odd and has only one positive root, then the system

$x'=y-F(x)$, $y'=x$, $x'=dx/dt$, has at least one periodic solution. The equation $x''+f(x)x'+x=0$ can be reduced to this system. The first proof is due to Lienard [Rev. Gén. Electr. **23**, 901 and 946 (1928)]; previous proofs (for special $F(x)$) were given by van der Pol, Greaves, E. and H. Cartan. The author shows that there exists a circle $\mathfrak{K}=E[x^2+y^2 \leqq \rho^2]$ and a domain Ω such that every integral curve which passes through a point $K \epsilon \mathfrak{K}$ will tend asymptotically to a closed curve $\mathfrak{u}_1$ and every integral curve passing through $Q \epsilon \Omega$ will tend asymptotically to a closed curve $\mathfrak{u}_2$. It follows from a theorem of Bendixon on integral curves that there exists at least one closed integral curve. This curve represents the periodic solution. *S. Bergmann.*

Adamoff, N. V. Sur quelques propriétés des transformations qui laissent invariable la courbe intégrale d'une équation du premier ordre. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 539–543 (1940). [MF 3967]

The operator $A(y)=y+\int_a^x X(x, y)[f(x, y)-y']dx$ leaves invariant the integrals of the differential equation $y'=f(x, y)$. The author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **18**, 219–223 and **19**, 17–22 (1938)] has given conditions on X and f such that $A(y_1)-y_1>0$ implies that the curve $y=y_1(x)$ lies below an integral curve of $y'=f$. The present paper treats this same problem under weaker conditions on X and f, and also the problem of finding sufficient conditions to insure that a curve $y=y_1(x)$ lies above an integral curve of $y'=f$. Applications of the results are made to an equation of the Riccati type. *F. G. Dressel* (Durham, N. C.).

Chiellini, Armando. Sugli invarianti delle equazioni differenziali del primo ordine a derivate ordinarie del tipo $y'=f(y)=c_0y^n+\binom{n}{1}c_1y^{n-1}+\binom{n}{2}c_2y^{n-2}+\cdots+\binom{n}{n-1}c_{n-1}y+c_n$. Pont. Acad. Sci. Comment. **4**, 385–411 (1940). [MF 4113]

Une transformation de la forme $y=\lambda(x)z(\xi)+\mu(x)$, $\xi=\phi(x)$ change l'équation donnée en une de la même forme en z et ξ. Par une telle transformation on peut éliminer le second et l'avant-dernier terme et réduire à l'unité le coefficient du premier. Les coefficients de l'équation transformée sont alors déterminés à moins de puissances d'une constante multiplicative arbitraire. L'auteur donne l'expression de n nombres "invariants relatifs" et de $n-1$ quotients formés avec eux "invariants absolus" (qui cependant dépendent encore de la dite constante et pourraient subir une petite modification pour rejoindre l'invariance absolue). L'auteur applique ces résultats pour donner les conditions pour qu'une équation du type considéré puisse se réduire, par une transformation de cette forme, aux coefficients constants et, par conséquent, s'intégrer par quadratures. *B. Levi* (Rosario).

González, Mario O. A general type of Lie's differential equations. J. Math. Phys. Mass. Inst. Tech. **20**, 80–88 (1941). [MF 3843]

Presents a systematic method of deducing, from a given type of differential equation invariant under a known group, new types of differential equations with known integrating factors. An unimportant error in theorem II is due to the erroneous statement in footnote 5. *P. Franklin.*

Tôyama, Hiraku. Some inequalities in the theory of linear differential equations. Tôhoku Math. J. **47**, 210–216 (1940). [MF 4022]

The following theorem for systems of ordinary, first-order, linear differential equations is established: Let $x(t) : (x_1, \cdots, x_n)$ be a vector and $A(t)=(a_{ij})$ be a square matrix of n rows. Let $|x|=[\sum_{i=1}^n |x_i(t)|^p]^{1/p}$ $(p>1)$, while $|A(t)|=\max_{|x|>0}[|A(t)x|/|x|]$. The solutions $x(t)$ of the system of differential equations

$$dx_i(t)/dt=\sum_{k=1}^n a_{ik}(t)x_k(t), \quad i=1, 2, \cdots, n,$$

on $t_0 \leqq t \leqq t_1$ satisfy the inequalities

$$\exp\left[\int_{t_0}^{t_1}|A(t)|dt\right] \geqq |x(t_1)|/|x(t_0)| \geqq \exp\left[-\int_{t_0}^{t_1}|A(t)|dt\right],$$

where the equalities hold when and only when

$$x(t)=\exp\left[\pm\int_{t_0}^{t}|A(t)|dt\right]x(t_0).$$

The proof of this theorem is based largely upon the approximation of solutions of the differential system through use of the Cauchy polygon method. *W. M. Whyburn.*

Hukuhara, Masuo. Théorèmes fondamentaux de la théorie des équations différentielles ordinaires, I. Mem. Fac. Sci. Kyūsyū Imp. Univ. A. **1**, 111–127 (1941). [MF 3976]

This paper develops fundamental existence, uniqueness and differentiation theorems for the ordinary differential equation $y'=f(x, y; \lambda)$, where x, y and f are real and λ is a parameter. The method of attack is quite similar to that introduced by Perron in defining the Perron integral and in proofs of existence theorems for ordinary and partial differential equations. The present author uses left and right over and under functions which are similar to the over functions and under functions used in the Perron work. The theorems obtained are of essentially the same generality as those obtained by others who have used the Perron approach. The developments of the paper are made with a special view to their ready adaptability to applications. It is quite likely that the author intends to follow the paper with another in which applications are given.

W. M. Whyburn (Los Angeles, Calif.).

Pougatcheff, W. Sur les expressions asymptotiques pour les intégrales des systèmes d'équations différentielles linéaires contenant un paramètre. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **5**, 75–84 (1941). (Russian. French summary) [MF 4271]

The author gives an extension of a part of G. D. Birkhoff's theory of linear differential equations containing a parameter. Under consideration is the matrix system (1) $dY(x)/dx=YP+\Pi$, where (2) $P \sim \alpha[P_0(x)+P_1(x)\alpha^{-1}+\cdots]$, $\Pi \sim \alpha^r Z[\Pi_0(x)+\Pi_1(x)\alpha^{-1}+\cdots]$; here the asymptotic relations are with respect to the parameter α, to $m(\geqq 1)$ terms, when α is in a certain region D extending to infinity, while x is on a closed interval (a, b); Z denotes a matrix solution of (3) $dZ/dx=ZQ$, where $Q \sim \alpha[Q_0(x)+\cdots]$; the P_k, Q_k, Π_k have derivatives of order $m-k+1$. It is assumed that the characteristic numbers $\lambda_i(x)$ of the matrix $P_0(x)$, and the $\mu_i(x)$ of $Q_0(x)$, are distinct for all x on (a, b), and that the differences $R(\alpha\lambda_i-\alpha\lambda_j)$, $R(\alpha\mu_i-\alpha\mu_j)$ do not change sign for α in D and x on (a, b); moreover, the values $R(\alpha\lambda_i-\alpha\mu_j)$ do not change sign, while the $\lambda_i-\mu_j$ are nowhere zero. It is proved that the system (1) (with $\Pi=0$) has a solution $Y \sim Z[Y_0(x)+Y_1(x)\alpha^{-1}+\cdots]$ (to m terms) and that the nonhomogeneous system (1) has a solution $Y \sim \alpha^{r-1}Z[Y_0(x)+Y_1(x)\alpha^{-1}+\cdots]$ (to m terms). This is

established on the basis of a lemma which is an immediate consequence of G. D. Birkhoff's theory.

W. J. Trjitzinsky (Princeton, N. J.).

Shifner, L. Again on the integration of the differential systems. Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **4**, 417–422 (1940). (Russian. English summary) [MF 4083]

The author considers the differential system (in matrix form) $Y^{(1)}=Y(U_1\varphi_1(x)+U_2\varphi_2(x))$ when $U_1U_2U_1=0$. Letting $f_n=\sum(-1)^s\binom{n}{s}U_2^{n-s}U_1U_2^s$ $(s=0,\cdots,n)$ and $L_i(b|x)=\int_b^x\varphi_i(t)dt$, $L_{i_1\cdots i_p}(b|x)=\int_b^x L_{i_1\cdots i_{p-1}}(b|t)\varphi_{i_p}(t)dt$, a matrix solution is obtained in the form $Y=\exp S(x)\cdot\exp(U_2L_2(b|x))$, provided $S(x)=U_1L_1(b|x)+f_1L_{2,1}(b|x)+f_2L_{2,2,1}(b|x)+\cdots$ converges. This generalizes an earlier result of I. A. Lappo-Danilevsky. *W. J. Trjitzinsky* (Princeton, N. J.).

van Kampen, E. R. Notes on systems of ordinary differential equations. Amer. J. Math. **63**, 371–376 (1941). [MF 4158]

The differential system $(x^i)'=f^i(x^1,\cdots,x^k;t)$ $(i=1,\cdots,k)$, where the functions f^i are defined and continuous over a $(k+1)$-dimensional domain which contains the origin, has at least one solution through the origin. Many conditions under which this solution is unique have been developed. One such condition has the form of the well-known Lipschitz condition, where the Lipschitz constant is replaced by the function $h(t)$ which is required to satisfy the inequality $h(t)\leqq a/t$, $a\leqq 1$. In case the above-mentioned condition is satisfied, it has been shown that, if $a<1$, then the unique solution of the differential system can be obtained by successive approximations. The present paper proves that the method of successive approximations will also yield the solution for the case where $a=1$. The paper also shows that uniqueness of the solution and convergence of the successive approximations are valid for the case where the function $h(t)$ is given by the equation $h(t)=g(t)+1/t$, where $g(t)$ is integrable. *W. M. Whyburn.*

Malmquist, J. Sur l'étude analytique des solutions d'un système d'équations différentielles dans le voisinage d'un point singulier d'indétermination. I. Acta Math. **73**, 87–129 (1940). [MF 3754]

This paper studies the differential system

$$x^{k+1}dy_i/dx=P_i(x,y_1,\cdots,y_n),\quad i=1,\cdots,n,$$

where k is a positive integer and the functions P_i are power series in their arguments which vanish at $x=0$, $y_1=0,\cdots,y_n=0$, and converge in a domain $|x|<r$, $|y_i|<r'$ $(i=1,\cdots,n)$. The general problem considered concerns the form of the domain of existence of a solution $y_1,\cdots,y_n$ and the analytic representation of this solution in this domain. The author has previously investigated the case of a single equation of the first order. The present paper investigates the general system under the hypothesis that the roots of the characteristic equation are simple and non-zero. A somewhat more general system where the functions P_i are regular for $0<|x|<r$, $|y_i|<r'$ $(i=1,\cdots,n)$, but are not necessarily uniform, is investigated. In this case, certain asymptotic conditions are developed and imposed on the functions. The existence of certain solutions of particular interest which satisfy inequalities of the form $|y_i|<\bar{r}_i$ $(i=1,\cdots,n)$ in a maximal sector is demonstrated and the results applied to the theory of irregular solutions of linear differential systems. *W. M. Whyburn* (Los Angeles, Calif.).

Malmquist, J. Sur l'étude analytique des solutions d'un système d'équations différentielles dans le voisinage d'un point singulier d'indétermination. II. Acta Math. **74**, 1–64 (1941). [MF 4223]

The present paper is a continuation of the paper reviewed above and treats the general cases where the characteristic equation may have multiple roots, one of which may be zero. Under quite general hypotheses, the system is transformed into a normal form which exhibits the terms of P_i which are linear in the y's. This normal system is studied. In cases where zero does not occur as a root of the characteristic equation, the results obtained are as complete as in the previous case of simple, non-zero roots. The results are less complete when a zero root occurs.

W. M. Whyburn (Los Angeles, Calif.).

Cohn, Richard. On the analogue for differential equations of the Hilbert-Netto theorem. Bull. Amer. Math. Soc. **47**, 268–270 (1941). [MF 4173]

It is known that, if $F_1,\cdots,F_r$ is a finite system of differential polynomials in the unknowns $y_1,\cdots,y_n$ and if G is a differential polynomial which vanishes for every solution of the system of differential equations $F_i=0$, $i=1,\cdots,r$, some power G^p of G is a linear combination of the F_i and their derivatives of various orders, the coefficients in the linear combination being differential polynomials. The previous proofs of this theorem are abstract and do not furnish a definite G^p as above. The author supplies a constructive proof in which a definite power of G, and a linear expression for it, are produced. *J. F. Ritt* (New York, N. Y.).

Bernstein, S. Première note sur les opérateurs différentiels linéaires. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 532–535 (1940). [MF 3965]

The author considers differential operators (1) $D_k(y)=\varphi_0(x)y^{(k)}+\cdots+\varphi_k(x)y$, where the $\varphi_i(x)$ are continuous. It is known that for every continuous $a(x)$ the equation (2) $D_k(y)=a(x)$ has solutions $y\subset C(k)$ (that is, with $y^{(k)}$ continuous) on every interval (a,b) where $\varphi_0(x)\geqq 0$. Thus, for every $\epsilon>0$, one can find polynomials $p_\epsilon(x)$ so that (3) $|D_k(p_\epsilon(x))-a(x)|<\epsilon$ on (a,b). Throughout, it is assumed that $\varphi_0(x)$, $\varphi_1(x)$ are not zero for some x. Complete proofs are intended to be given in another work. The author makes use of the following terminology. D_k is (S) on (a,b) if (3) holds. The singularities on (a,b) consist of the set of points E where $\varphi_0(x)=0$; y is a regular solution of (2) on (a,b) if $y\subset C(k-1)$. An isolated singularity x_0 is regular on the right (left) if all the solutions of (2) are regular on (x_0,x_0+h) for $h>0$ $(h<0)$, where $|h|$ is sufficiently small. Some of the results are as follows. D_k is (S) on (a,b) if and only if (2) has a regular solution on (a,b) for every continuous $a(x)$. In order that D_k $(k>1)$ be not (S) on (a,b) (1) it is necessary that there should exist two singularities $\alpha<\beta$ on (a,b) irregular (that is, not regular) on the right and left, respectively; (2) it is sufficient that there should exist $k+1$ singularities $\alpha<x_1<\cdots<x_{k-1}<\beta$, where $x_1,\cdots,x_{k-1}$ are irregular from both sides, while α and β are irregular from the right and left, respectively. The lines of investigation introduced by the author have important implications in the theory of linear differential equations.

W. J. Trjitzinsky (Princeton, N. J.).

Ascoli, Guido. Sulla decomposizione degli operatori differenziali lineari in fattori lineari e sopra alcune questioni geometriche che vi si riconnettono. Univ. Nac. Tucumán. Revista A. **1**, 189–215 (1940). [MF 4063]

The author establishes the results of Mammana [Math.

Z. 33, 186–231 (1931)] on the decomposition of linear homogeneous differential operators into the product of linear factors under hypotheses that are somewhat less restrictive than those assumed by Mammana. Of special interest are the geometric interpretations that the author gives for certain of his results. *W. T. Reid* (Chicago, Ill.).

Monna, A. F. On a system of differential equations with denumerably many unknowns. Nieuw Arch. Wiskde **20**, 291–300 (1940). (Dutch) [MF 3991]

The author proved in a previous paper [Nederl. Akad. Wetensch., Proc. **41**, 889–895 (1938)] that the infinite system of differential equations

$$d\varphi_n/dz=\Gamma_{n-1}\varphi_{n-1}-\Gamma_n\varphi_{n+1},\quad n=0,1,2,\cdots,$$

$\Gamma_{-1}=0$, Γ_n a uniformly bounded sequence, has unique solutions satisfying the conditions $\varphi_0(0)=1$, $\varphi_v(0)=0$, $v\geqq 1$; moreover $\sum_0^\infty \varphi_n^2(z)=1$. The Bessel functions satisfy the special system $J_n'=\frac{1}{2}J_{n-1}-\frac{1}{2}J_{n+1}$. In this paper it is proved with an additional assumption on the Γ_v that every function $f(z)$, regular in a neighborhood of $z=0$, has a unique development $f(z)=\sum_0^\infty \alpha_v\varphi_v(z)$. It is also shown that the Bessel functions are essentially the only system satisfying a recurrence formula of the usual type. *O. Szász.*

Burdette, A. C. On a local solution of a differential equation of infinite order. Amer. J. Math. **63**, 291–294 (1941). [MF 4151]

Let $F(z)=\sum_0^\infty a_n z^n$ be an entire function of exponential type not greater than $q<\infty$. The reviewer has shown [Duke Math. J. **3**, 593–609 (1937)] that constants α, β, γ exist such that, if $\phi(x)$ is analytic in a circle of radius $\lambda>\alpha$, then there is a function $y(x)$, analytic in $|x|<\beta$, that satisfies the linear differential equation of infinite order $L[y]\equiv\sum_0^\infty a_n y^{(n)}(x)=\phi(x)$ in $|x|<\gamma$. The constants α, β, γ were not sharply defined. By making the further hypothesis that $F(z)$ is bounded on a certain infinite sequence of expanding contours, the author finds more precise values for the constants: $\alpha=e^{q-1}$, $\beta=\log(\lambda e)$, $\gamma=\log(\lambda e^{1-q})$. *I. M. Sheffer.*

Molenaar, P. G. Ueber die Differentialkovariante erster Ordnung der binären kubischen Differentialform. Nederl. Akad. Wetensch., Proc. **44**, 41–46 (1941). [MF 4043]

The author applies the methods developed by E. Noether [R. Weitzenböck, Invariantentheorie, pp. 359–362] to a binary cubic differential form and thus obtains a linear differential covariant of the first order, previously determined in another manner by the author [Nederl. Akad. Wetensch., Proc. **41**, 278–288 (1939)]. *J. Williamson.*

Fantappiè, Luigi. Sulle soluzioni del problema di Cauchy per tutti i sistemi di equazioni a derivate parziali lineari e a coefficienti costanti d'ordine qualunque. Pont. Acad. Sci. Comment. **3**, 403–468 (1939). [MF 4103]

Étant donné un système de la forme

$$\frac{\partial z_r}{\partial x_0}+\sum_j\sum_k a_{rk}^{(j)}\frac{\partial z_r}{\partial x_j}+\sum_k a_{rk}^{(p+1)}z_k=f_r(x_0,x_1,\cdots,x_p),$$
$$j=1,2,\cdots,p;\ r,k=1,2,\cdots,\nu,$$

où les a sont des constantes et les f des fonctions analytiques, l'auteur établit une formule pour représenter la solution determinée par la condition de Cauchy que pour $x_0=x_0^0$ les z_r se reduisent aux fonctions analytiques données $\psi_r(x_1,\cdots,x_p)$. La dite représentation dépend de p intégrations ordinaires et de p intégrations sur des contours fermés respectivement sur les plans de Gauss de p variables complexes auxiliaires. Les fonctions à intégrer dépendent de certaines fonctions définies par des développements en séries de puissances des dites variables auxiliaires, dont les coefficients sont les éléments des matrices puissances de la matrice des coefficients du système donné. Comme on le voit le problème traité est beaucoup plus restreint que celui qui énonce le titre; il ne s'agit pas en effet de tous les systèmes linéaires à coefficients constants, mais seulement de ceux qui peuvent se réduire à la forme de Sofie Kowalewski; il ne s'agit pas d'équations d'ordre quelconque, mais de celles qui peuvent se transformer en systèmes du premier ordre dont le nombre des équations ne surpasse pas le nombre des fonctions inconnues; enfin les conditions de Cauchy sont considérées dans le cas le plus simple. Dans l'essence la solution donnée ne surpasse pas les résultats connus de Cauchy-Kowalewski; l'intérêt se concentre au contraire dans la forme concise de la formule finale et dans la méthode de démonstration. La première presente l'avantage de ne pas dépendre directement du domaine de convergence des séries de Taylor par lesquelles la solution se définit d'abord; la seconde présente une intéressante application de concepts de la théorie des fonctionnelles analytiques développée par l'auteur en d'autres travaux.
B. Levi (Rosario).

***Miller, Frederic H. Partial Differential Equations.** John Wiley & Sons, Inc., New York, 1941. ix+259 pp. $3.00.

List of contents: Ch. 1, Ordinary differential equations. Ch. 2, Partial differentiation and space geometry. Ch. 3, Origin of partial differential equations. Ch. 4, Linear equations of first order. Ch. 5, Non-linear equations of first order. Ch. 6, Fourier series. Ch. 7, Linear equations of second and higher orders. Ch. 8, Non-linear equations of second order. The exposition is purely formal. The book contains a considerable number of various special cases where partial equations can be solved, but the classification of second order equations into elliptic, hyperbolic and parabolic is not even mentioned. *J. D. Tamarkin.*

***Coulson, C. A. Waves. A mathematical account of the common types of wave motion.** Oliver and Boyd, Edinburgh; Interscience Publishers, Inc., New York, 1941. xii+156 pp. $1.50.

"The object of this book is to consider from an elementary standpoint as many different types of wave motion as possible. In almost every case the fundamental problem is the same, since it consists in solving the standard equation of wave motion; the various applications differ chiefly in the conditions imposed upon these solutions. For this reason it is desirable that the subject of waves should be treated as one whole, rather than in several distinct parts; the present tendency is in this direction." [Quotation from the preface.]

Contents: The equation of wave motion. Waves on strings. Waves in membranes. Longitudinal waves in bars and springs. Waves in liquids. Sound waves. Electric waves. General considerations.

Bourgin, D. G. The Dirichlet problem for the damped wave equation. Duke Math. J. **7**, 97–120 (1940). [MF 3392]

Extending the investigations of D. G. Bourgin and R. Duffin on the vibrating string equation [Bull. Amer.

Math. Soc. **45**, 851–858 (1939); these Rev. **1**, 120] the author treats the Dirichlet problem for the damped wave equation (1) $y_{xx}-y_{tt}+K^2y=0$ in the rectangle $R: 0\leqq x\leqq m\pi$, $0\leqq t\leqq T\pi$. The number of independent solutions is found to depend on the number of positive integers l, g which satisfy the Diophantine equation (A): $(lT)^2-(gm)^2=(mTK)^2$. This equation arises by substituting $y=\sin lx/m \sin gt/T$ into (1); it may be written also $\alpha l\rho(l)=g$ with $\alpha=Tm^{-1}$, $\beta=(Km)^2$, $\rho(l)^2=1-\beta l^{-2}$. The existence of solutions depends upon the degree of approximation of $\rho(l)$ and similar expressions by rational numbers. § 1 contains a comprehensive preparatory investigation of the Diophantine problem and of these approximations; it contains numerous results about the nature of α, β as rational or algebraic numbers. The main uniqueness theorem of the paper states that $y(x, t)\equiv 0$ is the only admissible solution (that is, satisfying suitable conditions of continuity) which vanishes on the boundary of R when and only when (A) has no integral solution. As an example of various related theorems, it may be mentioned that for rational α, β the admissible solutions which vanish on the boundary of R form at most a finite linear manifold. In order to construct solutions of the Dirichlet problem, the author applies Fourier expansions which he appraises by his approximation theorems. Thus he can state existence theorems of this type: if α is rational, β irrational, and if the assigned boundary values $y(x, 0)$, $y(x, T\pi)$ and $y(0, t)$, $y(m\pi, t)$ can be extended to all values of x and t, respectively, as odd functions of periods $2m\pi$ and $2T\pi$, respectively, such that they have sufficiently many continuous derivatives, then there exists a unique admissible solution. Additional discussions connect the Dirichlet problem with an associated Cauchy problem and state some explicit formulas. Finally the generalization to the case of n independent variables is discussed briefly. *E. D. Hellinger.*

Michlin, S. G. Problèmes aux limites fondamentaux de l'équation des ondes. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 281–285 (1940). [MF 3889]

This paper sketches a method for treating the mixed problem for $u_{xx}+u_{yy}-u_{zz}=0$, for a half cylinder with generators parallel to the z axis, determined by a simple analytic curve in the plane $z=0$. The generalized Green's theorem is used as in the classical Riemann method with the Volterra fundamental solution, and the region of integration is contained within the characteristic conoid and the half cylinder. The value of u at the vertex is expressed in terms of integrals involving the Cauchy data and an unknown function Q_σ; Q is to be determined by the condition of consistency with the Dirichlet or Neumann data on the half cylinder wall. For this purpose the vertex of the conoid is now taken on the cylinder, and after some transformations there is obtained an integrodifferential equation involving Q and Q_{zz}, with the domain of integration σ_0, the intersection on the half cylinder wall with the characteristic conoid. Following the usual pattern for the application of the Hadamard or Volterra generalization of the Riemann method a limiting process in conjunction with a certain natural class of approximating contours $\{\alpha^k\}$ is introduced. The definition and effective use of these preferred contours constitute the essential novelty of this paper. Eventually one arrives at an integral equation of Volterra type involving Q and both a double and a single integral; Q can now be determined by classical methods in terms of the data. There are a large number of misprints. *D. G. Bourgin* (Princeton, N. J.).

Weinstein, A. Les vibrations et le calcul des variations. Portugaliae Math. **2**, 36–55 (1941). [MF 4276]

This is an expository paper describing the use of the methods of the calculus of variations in the study of the vibrations of elastic plates under various boundary conditions. By way of introduction, the motions of a mechanical system with a finite number n of degrees of freedom in the neighborhood of a point of stable equilibrium are studied. It is pointed out that the frequencies of the n mutually orthogonal simple harmonic components of such motions are obtainable by a minimum principle as follows: the first (lowest) frequency is the minimum of one quadratic form where the second is equal to unity, the second frequency is the minimum of the first form with the second equal to unity, the points being restricted to the hyperplane "orthogonal" to the direction of the first motion, etc. It is noted that all the frequencies increase as linear restraints are imposed on the system and more detailed results along this line are derived.

The vibrations of two dimensional plates are studied and the problems are reduced to finding the values of λ for which the differential equation $\Delta\Delta w-\lambda^2 w=0$ has non-zero solutions and to finding the solutions. These are shown to be obtainable as the successive minima of one integral $(\int\int(\Delta w)^2 dxdy)$ when another $(\int\int w^2 dxdy)$ is held constant, the functions being subject to the proper boundary values ($w=0$ for the "supported" plate and $w=0$, $\partial w/\partial n=0$ for the "clamped" plate). It is shown that all the "characteristic" values (or frequencies) are increased when the stringency of the boundary conditions is increased. The value of the variational method as a tool for the computation of the frequencies, particularly those of the difficult clamped plate problem, and for the detection of errors in previous computing procedures is emphasized.

C. B. Morrey, Jr. (Berkeley, Calif.).

Pleijel, Åke. Propriétés asymptotiques des fonctions et valeurs propres de certains problèmes de vibrations. Ark. Mat. Astr. Fys. **27A**, no. 13, 101 pp. (1940). [MF 3492]

The asymptotic eigenvalue distribution for membranes, plates, thermic radiation and other problems significant in mathematical physics was first treated by H. Weyl on the basis of the theory of integral equations, then by R. Courant by variational methods. More recently, Carleman has established the connection between these problems and the theory of Tauberian theorems. Thereby in addition to theorems on asymptotic distribution of eigenvalues some information concerning the asymptotic behavior of eigenfunctions is also obtained.

The nucleus of this method is the following: Green's function of a one-parametric family of corresponding problems is expanded into a bilinear series by eigenfunctions; for example, in the membrane problem Green's functions for $\Delta w-k^2 w=0$. On the other hand the asymptotic behavior of Green's function for large k is evaluated directly. Then a Tauberian theorem permits the asymptotic description of terms or partial sums relating to the bilinear series. Pleijel's paper is a comprehensive presentation of this general idea carried out with many individual devices, overcoming considerable difficulties, and completing and simplifying the theory. In particular it should be mentioned that Pleijel achieves the necessary evaluation of the Green function by obtaining inequalities by variational considerations. The

"Tauberian" theory is applied in the form of a lemma by Hardy and Littlewood modified for the present purpose by the author. The paper covers the most important vibration problems. It deserves careful study as an original application of modern analytical methods to fundamental problems of classical physics. *R. Courant* (New York, N. Y.).

Carslaw, H. S. and Jaeger, J. C. The determination of Green's function for the equation of conduction of heat in cylindrical coordinates by the Laplace transformation. J. London Math. Soc. **15**, 273–281 (1940). [MF 4038]

The Green's function here represents the temperature at each point (r, θ, z), at time t, due to an instantaneous unit source of heat at an arbitrary point of the solid, at $t=0$, when the initial and surface temperatures are both zero. The function is determined for each of the following fifteen solids: $r<a$; $r>a$; $a<r<b$; $0<z<l$; $r<a$, $0<z<l$; $r>a$, $0<z<l$; $a<r<b$, $0<z<l$; $0<\theta<\theta_0$; $r<a$, $0<\theta<\theta_0$; $r>a$, $0<\theta<\theta_0$; $a<r<b$, $0<\theta<\theta_0$; $0<\theta<\theta_0$, $0<z<l$; $r<a$, $0<\theta<\theta_0$, $0<z<l$; $r>a$, $0<\theta<\theta_0$, $0<z<l$; $a<r<b$, $0<\theta<\theta_0$, $0<z<l$. The results are obtained by using the Laplace transformation in conjunction with series and integrals of normal forms of solutions of the transformed heat equation. Some of the less common properties of Bessel functions were also needed. *R. V. Churchill* (Ann Arbor, Mich.).

Carslaw, H. S. and Jaeger, J. C. The determination of Green's function for line sources for the equation of conduction of heat in cylindrical coordinates by the Laplace transformation. Philos. Mag. (7) **31**, 204–208 (1941). [MF 4236]

The Green's function here is the temperature $v(r, \theta, t)$ due to an instantaneous line source at $r=r'$, $\theta=\theta'$, $t=0$, in a solid with initial temperature zero and surface temperature zero. The authors use the Laplace transformation to derive the formula for the function v for each of the following seven solids: $r<a$; $a<r$; $a<r<b$; $0<\theta<\theta_0$; $0<\theta<\theta_0$, $r<a$; $0<\theta<\theta_0$, $a<r$; $0<\theta<\theta_0$, $a<r<b$. *R. V. Churchill.*

Copson, E. T. On an elementary solution of a partial differential equation of parabolic type. I. Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 37–53 (1941). [MF 4309]

For the parabolic equation $(k(x)u_x)_x - l(x)u = g(x)u_t$ the author writes an elementary solution in the form

$$\Gamma(x, \xi, t-\tau) = \sum_1^\infty y_n(x)y_n(\xi)e^{-\lambda_n(t-\tau)}, \quad a \leqq x; \ \xi \leqq b.$$

Here y_n is the normalized characteristic function which is associated with the nth characteristic number λ_n of the Sturm-Liouville system

$$\frac{d}{dx}\left\{k\frac{dy}{dx}\right\} + (\lambda g - l)y = 0, \qquad y(a) = y(b) = 0.$$

[See E. Rothe, Math. Z. **33**, 488–504 (1931), for a similar treatment of the parabolic equation with three space variables.] *F. G. Dressel* (Durham, N. C.).

Copson, E. T. On an elementary solution of a partial differential equation of parabolic type. II. The nature of the singularity. Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 54–60 (1941). [MF 4310]

It is shown that the singularity at $x=\xi$, $t=0$ of the solution $\Gamma(x, \xi, t)$ of the preceding review is similar to that of the elementary solution of the heat equation. *F. G. Dressel* (Durham, N. C.).

Feshbach, H. and Clogston, A. M. Perturbation of boundary conditions. Phys. Rev. (2) **59**, 189–194 (1941). [MF 3698]

The method of perturbations (that is, of expanding in powers of a parameter) is applied to the calculation of the variation of characteristic values and characteristic functions in cases where the boundary conditions are varied either through the variation of the boundary or where the boundary is fixed and the boundary conditions are changed. Applications of the method to problems in room acoustics, to electromagnetic resonators and to electronic wave functions for metals are mentioned. *H. Poritsky.*

Scherman, D. J. Une remarque concernant le problème de Dirichlet. C. R. (Doklady) Acad. Sci. URSS (N.S.) **29**, 286–287 (1940). [MF 3890]

If G is a bounded multiply connected region in three dimensions, it is in general impossible to express the solution of the Dirichlet problem for G as the potential of a double layer; in fact a single layer is required in addition to the double layer. If the boundary S of G consists of surfaces $S_1, S_2, \cdots, S_{m+1}$, where S_{m+1} contains the others, the author seeks to express the solution of the Dirichlet problem as the sum of the potentials of a double layer on S and of m point charges located at prescribed points Q_i inside $S_1, S_2, \cdots, S_m$, the amount of charge at Q_i being the same as the total strength of the double layer on S_i. The author, on expressing his potential in integral form, obtains an integral equation for $\nu(P)$, the density of the double layer, and he is able to show that the associated homogeneous equation has only the trivial solution. Hence his integral equation is solvable and the solution to the Dirichlet problem is expressible in the desired fashion. *J. W. Green* (Rochester, N. Y.).

Chwedelidse, B. Über die Poincarésche Randwertaufgabe des logarithmischen Potentials. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 195–198 (1941). [MF 4265]

If T is a bounded domain with a closed simple curve L as frontier, the problem of finding a harmonic function u in T which satisfies on L the relation (1) $du/dn + p(s)du/ds + q(s)\cdot u + f(s) = 0$, where p, q and s are given functions, is called Poincaré's boundary value problem. It need not have a solution even if p, q, f and L are analytic. The author states that he is able to solve the problem completely if p, q, f are real and satisfy a Hölder condition and L has a continuous curvature. The integral equation which results when u is expressed as the potential of a single layer has a divergent integral which must be taken in the sense of Cauchy's principal value. By the use of an identity of Vecoua, this singular integral equation is transformed into a non-singular Fredholm equation, to which the author claims it is equivalent. Since the existence of solutions of the Fredholm equation is not discussed, it is not clear in exactly what way the paper solves the given boundary value problem. In fact, it is stated that in order that the given boundary value problem be equivalent to the Fredholm equation, it is necessary to assume certain conditions on du/dn, which then assumes that u exists. In particular, no necessary conditions of an integral nature on p, q and f are given, which would seem to be required. The author also states without proof that, if any solution of (1) can be shown a priori to be unique if it exists, then the Fredholm equation has a solution which also solves (1). *J. W. Green.*

Martin, Robert S. Minimal positive harmonic functions. Trans. Amer. Math. Soc. **49**, 137–172 (1941). [MF 3670]

A positive harmonic function in a domain D is called

minimal if it dominates there no positive harmonic functions except its own constant submultiples. The object of the paper is to investigate the possibility of representing all positive harmonic functions $u(P)$ in D in the form $\int_\delta f(S, P)d\mu(e_\delta)$, where $\mu(e)$ is a positive mass distribution on the boundary d of D, depending on $u(P)$, and the $f(S, P)$ are a family of functions, called a basis, harmonic in P for P in D and S on d, depending only on the domain D. Two main questions arise and are answered affirmatively in the paper, with suitable qualifications: Is the class of minimal harmonic functions in D sufficiently wide to form a basis, and if so, is the resulting representation unique?

If $G(M, P)$ is the Green's function of D, the author defines $K(M, P)=G(M, P)/G(M, P_0)$, where P_0 is a fixed point. If, for a sequence $\{M_n\}$ of points of D not accumulating in D, $\{K(M_n, P)\}$ converges to a harmonic function, denoted by $K(M, P)$, the sequence $\{M_n\}$ is said to define (or be) an ideal boundary element M of D. Two ideal boundary elements are called identical if the defining sequences converge to the same harmonic function. The set Δ of ideal boundary elements is adjoined to D to obtain a space $\mathfrak{D}$ with every point of which is associated a $K(M, P)$. With the metric

$$\rho(M, M')=\int_\Sigma \frac{|K(M, P)-K(M', P)|}{1+|K(M, P)-K(M', P)|}dP,$$

where Σ is a small sphere about P_0, $\mathfrak{D}$ can be shown to be complete and compact, and the part D, which is open, has the same topology as it had with the original metric. If $u(P)$ is harmonic and positive, then a distribution $\mu(e)$ on Δ exists such that $u(P)=\int_\Delta K(M, P)d\mu(e_M)$. The author now shows that every minimal harmonic function in D is a constant multiple of some $K(M, P)$, and, if Δ_1 is the subset of Δ where $K(M, P)$ is minimal, then there exists a mass $\mu_1(e)$, actually distributed on Δ_1, such that $u(P)=\int_{\Delta_1}K(M, P)d\mu_1(e_M)$. This representation is called canonical, makes use of only the minimal functions for its basis, and is unique. In case the subset Δ_0 of Δ where $K(M, P)$ is not minimal is not void, then the representation $u(P)=\int_\Delta K(M, P)d\mu(e_M)$ is in general not unique.

The paper closes with several examples showing that the difficulties encountered in the integral representation are genuine and not due merely to weakness of method. In particular, an example is exhibited of a topologically simple domain for which Δ_0 is not void, and also an example where a continuum of ideal boundary elements is located at a single boundary point. *J. W. Green* (Rochester, N. Y.).

Monna, A. F. Sur un principe de variation de Gauss dans la théorie du potentiel. Nederl. Akad. Wetensch., Proc. **44**, 50–61 (1941). [MF 4045]

Given a positive mass distribution of potential V on an open set Ω of bounded frontier Σ, with $F=\Omega+\Sigma$, the distribution $\underline{\mu}(e)$ is obtained by the sweeping out process, the distribution $\bar{\mu}(e)$ by the process of extremalization (Brélot). If $\rho(e)$ is any distribution of positive mass on Σ such that its potential U satisfies the conditions $U\leqq V$ everywhere and $U=V$ on CF, then U satisfies the inequality $\text{Pot}\,\underline{\mu}\geqq U\geqq\text{Pot}\,\bar{\mu}$, the extremes of the inequality not being identical necessarily on F. The author discusses the evaluation of $\rho(e)$ itself in terms of $\bar{\mu}(e)$ and $\underline{\mu}(e)$ by means of Stieltjes integrals whose integrands $\xi(Q)$ and $1-\xi(Q)$ depend on the given non-negative additive function of point sets $\rho(e)$ but not on e. The demonstration is not complete. The second main theorem is that the mutual energy integral $\int(U-2V)d\rho$ of such distributions $\rho(e)$ is a maximum when $\rho(e)$ is identical with $\bar{\mu}(e)$. This result is applied to the energy of distributions lying on Σ whose potential is constant in Ω (when Ω is bounded) and elsewhere less or equal to that constant. The author points to generalizations of these theorems for fractional potentials $1<\alpha<3$. *G. C. Evans.*

Wavre, R. Sur l'identification des potentiels. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 87–94 (1940). [MF 4407]

The author considers potentials of analytic single and double layers and volume distributions on analytic bodies, that is, bodies bounded by analytic surfaces, and necessary and sufficient conditions that a given function Φ be the potential of such a system of distributions. The conditions appear in terms of the single valuedness and analyticity of the functions obtained by extending Φ across the given surfaces, and of the behavior of Φ at ∞. *J. W. Green.*

Gelfand, I. Die direkte und umgekehrte Aufgabe der Bestimmung des Anziehungspotentials eines homogenen Kreissegments. Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 89–94 (1941). (Russian. German summary) [MF 4274]

Die Arbeit beschäftigt sich mit der Berechnung des Potentials für ein Kreissegment. Es wird eine Methode zur Verarbeitung der Gravitationsbeobachtungen nach den Werten des Gradienten der Schwerkraft und der Krümmungsdifferenz, auf Grund der Lösung der direkten Aufgabe, angegeben. *Author's summary.*

Diatschenko, W. und Breus, K. Das Potentialfeld eines Diaphragmas. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 5, 3–16 (1940). (Ukrainian. Russian and German summaries) [MF 3748]

The problem indicated in the title can be reduced to the determination of an harmonic function $w(z, r, \phi)$ which assumes on the annulus $E[z=0,\ a\leqq r\leqq C]$ the constant value w_0 and for which $\lim_{R\to\infty} Rw=e$ holds. Using the relation $z+ir=ia\ \text{sn am}\,(u+iv|k)$, the authors introduce new variables u, v. The potential equation becomes $\partial^2\psi/\partial u^2+\partial^2\psi/\partial v^2+\frac{1}{4}\mu^2\psi=0$, where $\psi=wr^{\frac{1}{2}}$ and $\mu^2=(\partial\lg r/\partial u)^2+(\partial\lg r/\partial v)^2$. Separating the variables, that is, putting $\psi=\psi_1(u)\cdot\psi_2(v)$, they obtain ordinary differential equations of the Mathieu type for ψ_2. The characteristic values and characteristic functions of these equations can be determined by successive approximations. Using these results the authors give formulas for w. *S. Bergmann.*

Blanc, Charles. Les réseaux Riemanniens. Comment. Math. Helv. **13**, 54–67 (1940). [MF 3988]

Functions of discrete points, defined at the vertices of a plane network, are again considered [cf. Blanc, Comment. Math. Helv. **12**, 153–163 (1939–40); these Rev. **1**, 213]. A function is said to be harmonic at P if its value there is equal to the average of the values at neighboring points. Conjugate harmonic functions are defined and certain results analogous to theorems concerning ordinary harmonic functions and analytic functions of a complex variable are obtained. Derivative, integral, type and curvature are defined; and there are a Green's formula, a Gauss-Bonnet formula, criteria concerning the type of a net and an isoperimetric inequality. *E. F. Beckenbach* (Ann Arbor, Mich.).

GEOMETRY

★**Sohon, F. W. The Stereographic Projection.** Chemical Publishing Co., Inc., New York, 1941. ix+210 pp. $4.00.

Exhaustive treatment of the subject with special emphasis on the treatment by numerical and nomographical means. Extensive tables are given, especially for use in seismological problems.

O. Neugebauer (Providence, R. I.).

Watson, G. N. An inequality in the triangle. Quart. J. Math., Oxford Ser. **11**, 273–276 (1940). [MF 3789]

Let ABC be any triangle and let P be any point inside it; let AP, BP, CP produced meet the opposite sides of the triangle at A', B', C'. Then at least one of the inequalities $AA' : BC \geqq \sqrt{3}/2$, $BB' : CA \geqq \sqrt{3}/2$, $CC' : AB \geqq \sqrt{3}/2$ holds. More precisely: That one of the above inequalities which involves the shortest of the three sides of the triangle is true. Or otherwise: That one of the above inequalities which involves the longest of the three transversals AA', BB', CC' is true. The author proves these propositions analytically, and offers also a geometric proof of the first proposition.

N. A. Court (Norman, Okla.).

Banning, J. On a generalization of F. Morley's theorem. Mathematica, Zutphen. B. **9**, 17–33 (1940). (Dutch) [MF 4426]

Wichers, J. On the hypercycloid of Steiner-Schläfli and its connection with Morley's triangle. Mathematica, Zutphen. B. **9**, 114–120 (1941). (Dutch) [MF 4049]

Baer, W. S. J. Über dreiseitige Pyramiden mit vier gleichen Höhen. Tôhoku Math. J. **47**, 261–278 (1940). [MF 4027]

The author derives a number of properties of the isosceles tetrahedron, but little, if anything, of what he says is new. Cf., for instance, Nathan Altshiller-Court, Modern Pure Solid Geometry, Macmillan, New York, 1935, pp. 94–101. See also p. 300 for references to original sources.

N. A. Court (Norman, Okla.).

Dehn, Max. Bogen und Sehnen im Kreis, Paare von Grössensystemen. Norske Vid. Selsk. Forh. **13**, 103–106 (1940). [MF 3869]

Hofmann, Jos. E. Bestimmung des Zylinderdurchmessers durch eine Oberflächenkonstruktion mit Zirkel und Lineal. Deutsche Math. **5**, 401–404 (1941). [MF 4252]

Sarmento de Beires, R. Sur la formule d'Euler-Savary. Portugaliae Math. **2**, 81–84 (1941). [MF 4278]

The so-called formula of Euler-Savary of plane kinematics and its extension to the displacement of a sphere in itself are derived, in a rather simple way, using the projective properties of a complete quadrilateral. *E. Helly.*

Fraile, V., Fraile, A. and Crespo, C. The locus and the loci of point domains in the plane. Revista Union Mat. Argentina **7**, 45–50 (1941). (Spanish) [MF 4458]

Little, Neil. An analytical study of perspective drawings of quadric surfaces. Amer. Math. Monthly **48**, 175–180 (1941). [MF 4132]

The perspective drawing of quadratic surfaces, that is, their projection from a point on a plane, is investigated analytically, using rectangular coordinates and the methods of elementary analytic geometry. *E. Helly.*

Krames, Josef. Über Helligkeitskonstruktionen auf experimenteller Grundlage und einige Bemerkungen über "graphische Funktionen." Monatsh. Math. Phys. **49**, 279–294 (1940). [MF 3092]

There are several formulas for determining the brightness of the elements of a surface. The formula for the construction of "isophotals" (loci of points of equal brightness) which is commonly used in descriptive geometry was given by Lambert, and identifies brightness and illumination. Extensive measurements of the brightness as a function of the direction of the incident and the reflected ray of light have been carried out by C. Wiener [Festgabe der technischen Hochschule, Karlsruhe, 1892]. The purpose of the present paper is to use Wiener's results for the construction of isophotals. The author gives no simple analytic representation of Wiener's results, but finds an empirical graphic method, which is only a little more complicated than the usual methods referred to above. His construction gives a close approximation to the results of Wiener's experiments.

E. Helly (Paterson, N. J.).

Blaschke, Wilhelm. Zur analytischen Geometrie in der Ebene von Hermite. Mitt. Math. Ges. Hamburg **8**, part 2, 3–30 (1940). [MF 4114]

[For the concept of geometry in the Hermitian plane, cf. Blaschke and Terheggen, Rend. Sem. Mat. Univ. Roma **3**, 153–161 (1939); these Rev. **1**, 261.] A point, or vector, is given by its three homogeneous complex coordinates x_1, x_2, x_3; the scalar product of two points is $\sum x_k \bar{y}_k$, and a displacement is a linear transformation which leaves the scalar products invariant; it is known as a unitary transformation.

The object of the investigation is first the "normal chain N_{I}," or the system of all points $\mathfrak{p} = \mathfrak{x} \cos\theta + \mathfrak{y} \sin\theta$, where $\mathfrak{x}$ and $\mathfrak{y}$ are two normed vectors, perpendicular to each other; secondly, the "normal chain N_{II}," or the system of all points $\mathfrak{p} = \xi\mathfrak{x} + \eta\mathfrak{y} + \zeta\mathfrak{z}$, where $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ are three normed points, perpendicular to one another, and the three real scalars ξ, η, ζ satisfy the equation $\xi^2 + \eta^2 + \zeta^2 = 1$. The problem is to find the invariants of the simplest combinations of points, normal chains N_{I} and normal chains N_{II}. The complete systems of invariants and their geometrical interpretation in terms of intersection, incidence and distance is given for figures formed by a point and a normal chain N_{I}, two normal chains N_{I}, N_{I}, or N_{I}, N_{II}, or N_{II} and N_{II}. If, in the definition of the normal chains, the condition of orthogonality of the defining points is omitted, the more general concept of chain K_{I} or K_{II}, already used by Staudt, is obtained. Blaschke proves that a chain K_{I} as well as a chain K_{II} has only one invariant, and that these invariants can be chosen so that their vanishing is the condition for the normality of the chains. The author then examines the projective geometry of chains and states that in it the theory of the invariants of two chains can be developed by means of elementary divisors. *E. Helly* (Paterson, N. J.).

Terheggen, Hans. Ein- und zweiparametrige Bewegungsvorgänge starrer Körper im Euklidischen R_3 und ihr Zusammenhang mit der Kurven- und Flächentheorie in einer M_6^2 des quasielliptischen R_7 der gebundenen Biquaternionen. Mitt. Math. Ges. Hamburg **8**, part 2, 105–146 (1940). [MF 4117]

The paper starts with a short survey of the well-known connections between biquaternions and the theory of dis-

placements in an Euclidean three-space. In this case, a biquaternion is defined as an expression $a+\epsilon\alpha$, where a and α are quaternions and ϵ the dual unity, for which $\epsilon^2=0$. There is a mapping of the system of the ∞^6 displacements in the system of the ∞^6 biquaternions, whose coefficients satisfy two quadratic equations. The author examines systematically the differential geometry of curves and surfaces (systems of ∞^1 and ∞^2 elements) in the biquaternion space, and shows that in this way a working method can be found for studying systems of displacements depending on one or two parameters. *E. Helly* (Paterson, N. J.).

Bottema, O. On associated lines in S_4. Nederl. Akad. Wetensch., Proc. **43**, 1282–1287 (1940). [MF 3941]

There is a well-known figure in four-dimensional space consisting of five associated straight lines, each of which is met by all planes meeting the four other lines. van der Woude [Nederl. Akad. Wetensch., Proc. **43**, 946–954 (1940); cf. these Rev. **2**, 153] examined the case where four of these lines l_1, l_2, l_3, l_4 have a common transversal l, and proved a few theorems concerning this figure, among them the fact that in this case there are ∞^1 lines l_5 associated with l_1, l_2, l_3, l_4 and that these associated lines build up a surface V_2^3 of dimension two and degree three. The author simplifies these investigations, gives a geometrical construction for the V_2^3, which he shows to be connected with Veronese's surface, and discusses the group of collineations which leaves the associated lines invariant. *E. Helly* (Paterson, N. J.).

Salzert, Margarete. Die Eigenschaften derjenigen Kollineationen die zwei konjugiert imaginäre windschiefe Geraden im Raume festlassen. Schr. Math. Inst. u. Inst. Angew. Math. Univ. Berlin **5**, 133–177 (1940). [MF 3486]

A collineation which leaves fixed two conjugate imaginary non-intersecting lines λ_1, λ_2 may be considered with reference to the congruence of lines intersecting λ_1, λ_2, or with reference to the quadric surfaces F_2 containing λ_1, λ_2. If the equations of λ_1, λ_2 are taken to be

$$z_1\equiv\xi_0+i\xi_3=0,\quad z_3\equiv\xi_0-i\xi_3=0,$$
$$z_2\equiv\xi_2+i\xi_1=0,\quad z_4\equiv\xi_2-i\xi_1=0,$$

then any line of the congruence will intersect the plane $\xi_0=0$ in a point $(0, \xi_{01}, \xi_{02}, \xi_{03})$, and writing

$$Z=\frac{z_2}{z_1}=\frac{\xi_2+i\xi_1}{\xi_0+i\xi_3}=\frac{\xi_{01}-i\xi_{02}}{\xi_{03}},$$

there is established a correspondence between the lines of the congruence in the ξ-space and the points of the z-plane $\xi_0=0$. The equation of any F_2 containing λ_1, λ_2 is of the form $x_0(\xi_0^2+\xi_3^2)+x_3(\xi_2^2+\xi_1^2)-2x_1(\xi_0\xi_2+\xi_1\xi_3)-2x_2(\xi_2\xi_3-\xi_0\xi_1)=0$, which is degenerate if $x_1^2+x_2^2-x_0x_3=0$. The numbers x_0, x_1, x_2, x_3 are taken as the coordinates of F_2, which may thus be represented by a point in the x-space.

Having set up this machinery in part I, the author proceeds in part II to discuss the group of collineations in the ξ-space which leaves λ_1, λ_2 fixed; she concentrates her attention upon the invariants, both geometrical and numerical, discussing them with reference to the ξ-space as well as the z-plane and the x-space. In part III the problem is considered from the point of view of differential geometry, and the invariants of various families of quadrics and of curves related to the congruence of lines are determined.

G. de B. Robinson (Toronto, Ont.).

Fano, Gino. Su alcune particolari reti di quadriche dello spazio ordinario. Univ. Nac. Tucumán. Revista A. **1**, 271–281 (1940). [MF 4066]

In Comment. Math. Helv. **12**, 172–190 (1940) [these Rev. **1**, 266] the present author considers the ∞^4 system of plane quartic curves tangent to a given quintic in 10 points. If these 10 points lie on a cubic, the system is called the first type; if not, the second. The present paper considers the finite number of quartics in each type consisting of a pair of conics. Each conic may touch the quintic in 5 points, and the points of intersection of the conics not belong to the quintic, or each conic may touch the quintic in 3 points, and the points of intersection belong to the quintic. Systems of the second type can be constructed by projecting a general net of quadric primals of S_4, the cones of which form a general quintic [W. P. Milne, J. London Math. Soc. **2**, 79 (1927) and W. L. Edge, Acta Math. **64**, 185–242 (1935)]. The vertices of the cones in the net form a curve of order 10 and genus 6, birationally equivalent to the quintic. The configuration was obtained by W. L. Edge [Proc. London Math. Soc. (2) **41**, 337–360 (1936)], who interchanges the first and second types. Various special cases are considered, in particular those in which one or both the conics may be composite. *V. Snyder* (Ithaca, N. Y.).

Cunningham, A. B. Non-involutorial space transformations associated with a $Q_{1,2}$ congruence. Bull. Amer. Math. Soc. **47**, 309–312 (1941). [MF 4183]

The paper is an extension of that of De Paolis [Atti Accad. Naz. Lincei. Rend. (4) **1**, 735–742, 754–758 (1885)] on transformations belonging to a line congruence meeting a curve of order m and an $(m-1)$-fold secant line. Given a conic r, a line s meeting r once, and two projective pencils of surfaces $|F_{n+m+1}|:r^n s^m g$, $|F'_{n'+m'+1}|:r^{n'}s^{m'}g'$, $n\leqq m+1$. A point P fixes a surface of $|F|$, and the unique line through P, r, s meets F' in one residual point P', image of P in (T). The complete Cremona table of this transformation is constructed for each of the three possible cases.

V. Snyder (Ithaca, N. Y.).

Perl, Andreas. Die singulären Punkte bei algebraischen Kurven von höchstens der 6. Ordnung. Mitt. Math. Ges. Hamburg **8**, part 2, 152–163 (1940). [MF 4119]

The singularities of plane quintics and sextics are systematically studied beginning with types of singularities of order 2 and continuing up to those of order 5. The treatment is entirely algebraic. *T. R. Hollcroft* (Aurora, N. Y.).

Cassity, C. Ronald. On the quartic Del Pezzo surface. Amer. J. Math. **63**, 256–262 (1941). [MF 4148]

Let P_ν $(\nu=0, 1, 2, 3, 4)$ be five points of general position in a plane. Let $f_\alpha(x_0, x_1, x_2)=0$ be the (cubic) locus of points of contact of tangents from P_α to conics through the other four P's. The author shows that every four of the five expressions f_α^2 are connected by a linear identity, and that only two of these five identities are linearly independent. It follows that the parametric equations

$$\rho y_\alpha=f_\alpha(x_0, x_1, x_2),\qquad \alpha=0, 1, 2, 3, 4,$$

define the surface of intersection of two quadric hypersurfaces in four dimensions, referred to the common self-polar simplex of the quadrics. Then, restricting consideration to the case when the 16 lines on the surface are all real, he observes that these lines form a semi-regular map dividing

the surface into 16 pentagonal and 20 quadrangular regions, which are synclastic and anticlastic, respectively.

H. S. M. Coxeter (Toronto, Ont.).

Enriques, Federigo. Sopra le involuzioni irregolari appartenenti ad una superfice algebrica. Univ. Nac. Tucumán. Revista A. **1**, 293–296 (1940). [MF 4068]

The author proves the theorem: If an algebraic surface F of genera p_a and p_g and irregularity $d=p_g-p_a>0$ contains a single infinity of involutions I of order n, genera p_a', $p_g'\leqq p_g$, and of the same irregularity $d=p_g'-p_a'$ as F, then F contains an irrational pencil of curves of genus d. In the proof of the above theorem the following theorem is established: The groups of curves of an ∞^1 series on F passing through two conjugate points of the involution I (for example, P_1 and any one of the conjugate points $P_2, \cdots, P_n$) are equivalent. Specifically, it is shown that those points P of F, through which the groups of curves of a continuous ∞^1 series $|C|$ on F are equivalent to the groups of curves of $|C|$ through the points P' conjugate to P in the involution I, lie on algebraic curves K which form on F an irrational pencil of genus d.

T. R. Hollcroft (Aurora, N. Y.).

Thrall, Robert M. On projective equivalence of trilinear forms. Ann. of Math. (2) **42**, 469–485 (1941). [MF 4296]

The two fundamental problems in the projective classification of trilinear forms are (I) the determination of necessary and sufficient conditions for the (projective) equivalence of two trilinear forms; (II) the determination of a set of forms which consists of one and just one representative form from each class of equivalent forms. The present paper considers the case in which two of the forms y, z are ternary and the third, x, quaternary in the field of complex numbers. Methods similar to those of Thrall and Chanler for the (3, 3, 3) case [Duke Math. J. **4**, 678–690 (1938)] are employed, using the notation and various geometric results of Room [Geometry of Determinantal Loci, Cambridge, England, 1938]. Two points y, z are conjugate if their coordinates simultaneously cause all coordinates of x to vanish, and similarly for the other pairs. These sets of equations define Cremona transformations, or the map of a manifold on a prime, or singular transformations. Two general trilinear forms are projectively equivalent if and only if their associated principal manifolds are projectively equivalent. Corresponding theorems are given for the special, nondegenerate linear forms, for the nodal and degenerate cases, including the complete enumeration of the 53 cases of the latter, which include both forms of proper ruled surfaces. Incidentally the following theorem is proven. Not every quaternary cubic form can be written as a three rowed determinant whose elements are linear forms. The exception may be expressed by $x_1^3+x_2^2x_3+x_3^2x_4$.

V. Snyder (Ithaca, N. Y.).

Severi, Francesco. Caratterizzazione topologica delle superficie razionali e delle rigate. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 51–60 (1940). [MF 4404]

An algebraic curve is fixed topologically by its genus, which defines its Riemann surface; it is fixed birationally by its genus and its moduli. A rational curve, having no moduli, is fixed by its Riemann surface (plane or sphere). It is not yet known just what attributes fix a surface or variety topologically. The base involves both algebraic-geometric and arithmetic attributes [Severi, Mem. Accad. Naz. Italia **5**, 239–283 (1934)].

The present paper obtains the criteria for rational surfaces and for those surfaces reducible to ruled surfaces, rational or irrational. The necessary and sufficient conditions that all the two dimensional cycles of a surface are algebraic is $p_g=0$ [see also Bagnera and DeFranchis, Rend. Circ. Mat. Palermo **30**, 185–238 (1910); see p. 230]. If F is a regular surface with $p_g=0$, it has a curve C (effective or virtual) for base. If C is effective, F is rational; if C is virtual, its adjoint $C'=lC$, in which l is -2, and F rational, or $l=4$. The latter may not be rational. The Riemannian of a plane is characterized by having linear connection and torsion both zero, two dimensional connection 1, and every two dimensional cycle connected [Severi, Ist. Lombardo, Rend. (2) **38**, 859–865 (1905)]. It is not fixed by having linear connection zero and a two dimensional cycle simply connected and of virtual grade 1. The Riemannian of an algebraic surface birationally reducible to a ruled surface of genus $p>1$ is characterized by having linear connection $2p$ and superficial connection 2. The torsion is then necessarily zero. For $p=1$, the superficial cycle must also be simply connected.

V. Snyder (Ithaca, N. Y.).

★**Hodge, W. V. D. The Theory and Applications of Harmonic Integrals.** Cambridge University Press, Cambridge, England; Macmillan Company, New York, 1941. ix+281 pp. $4.50.

The theory of harmonic integrals on a Riemannian manifold was developed by Hodge in the course of his investigations on the integrals attached to an algebraic variety. It is possible indeed to put a finger on the exact spot of the transcendental theory of algebraic varieties which, in last analysis, is responsible for the appearance of this stimulating book: it is the question of whether or not there exist double integrals on an algebraic surface (or, more generally, m-fold integrals on an m-dimensional algebraic variety) which are everywhere finite and which have all their periods equal to zero. This question remained unsolved for a long time until Hodge proved that such integrals do not exist (subsequently it has been recognized that a similar theorem holds, more generally, for integrals of closed forms on a topological manifold, as a corollary of results obtained previously by de Rham). However, the new theory, although an outgrowth of specific problems in algebraic geometry, goes far beyond algebraic geometry proper. Indeed a glance at the table of contents shows that of the five chapters which make up the book, only one deals with algebraic varieties. The rest is an ingenious blend of analysis pure and simple, topology, tensor calculus, differential geometry and continuous groups. The book therefore should also prove of considerable interest to specialists in the fields just mentioned. Since the original investigations of Hodge are scattered in many notes in which the topic is at times presented in a somewhat condensed fashion, the present book will be welcome by those who wish to gain a better understanding of the central points of the theory of harmonic integrals. The book is exceedingly well written, there is great emphasis on rigor, and the original material is presented anew in a systematic and unified form.

The first chapter, "Riemannian manifolds," is introductory and presents the material from differential geometry and topology which is needed later and used extensively throughout the book. The presentation of this auxiliary material is so well done that the reader has no need of

consulting the textbooks given in the references, unless he wants details of some proofs. From the topological material, the intersection theory of Lefschetz and the properties of product manifolds play a specially important role later on.

The second chapter, "Integrals and their periods," deals with the theory of integrals on a topological manifold, due to de Rham. Here we find the main properties of closed and null differential forms and the following two theorems of de Rham: (1) The periods of an integral of a closed p-form on the R_p independent p-cycles of the manifold can be assigned arbitrarily. (2) If the periods are all zero then the closed form is necessarily a null form; in other words, the periods determine the closed form uniquely, to within an arbitrary null form. In the case of algebraic integrals on an algebraic variety, this second theorem is the original theorem of Hodge.

From the standpoint of the general theory, the most important chapter is the next one, entitled "Harmonic integrals." As to degree of generality, the notion of an harmonic integral lies somewhere between the general notion of an integral on a topological manifold, developed in the previous chapter, and the notion of an algebraic integral on an algebraic variety discussed in the next chapter. To define an harmonic integral we need not only a topology but also a Riemannian metric on the manifold. The harmonic integrals are then special integrals satisfying certain conditions relative to the given Riemannian metric. On the other hand, the real and imaginary parts of an algebraic integral on an algebraic variety are harmonic integrals with respect to suitable metrics which can be introduced on the variety. This is not the place to go into the details of the definition of an harmonic integral. Suffice it to say that a closed p-form is defined to be harmonic if a certain dual $(n-p)$-form (n is the dimension of the manifold) is also closed. For example, in the case $n=2$, the linear element being given by $ds^2=\lambda(dx^2+dy^2)$, the dual of the form $Pdx-Qdy$ is $Qdx+Pdy$, and the forms are harmonic if they are both closed, that is, if $\partial P/\partial y+\partial Q/\partial x=\partial P/\partial x-\partial Q/\partial y=0$. In this case the forms are the real and imaginary part of an analytic integral. The main theorem proved in this chapter is the existence theorem for harmonic integrals with preassigned periods. The proof requires a deep analysis and makes use of the theory of Fredholm integral equations. In the course of the proof it is shown directly that if the periods of an harmonic integral are all zero then the differential form is identically zero. From this the second theorem of de Rham follows readily.

The fourth chapter, "Applications to algebraic varieties," contains some of the most interesting results of the author. We have already mentioned that the real and imaginary part of an algebraic integral are harmonic integrals. However, not every harmonic integral arises from an algebraic integral. Moreover, an harmonic integral on an algebraic variety is not a birationally invariant concept, since there are infinitely many ways of defining a metric on the variety. Nevertheless, from the properties of harmonic integrals it is possible to extract properties which are birationally invariant, either in the absolute sense, or in the more restricted, relative sense. It is this birational aspect of the theory of harmonic integrals that is emphasized in this chapter. If the variety V_m lies in a projective space S_r, there is a metric on V_m induced by the representation of S_r as a real analytic locus of $2r$ dimensions, due to Mannoury. It is necessary to assume that V_m is free from singularities in order to have a topological manifold. The fact that the $2m$ real coordinates of V_m are the real and imaginary parts of m complex uniformizing parameters implies certain special properties for the metric which play a role in the proofs. For instance, if $T^{i_1\cdots i_q}_{j_1\cdots j_p}$ is a tensor, then also $T^{i_1\cdots i_q}_{j_1\cdots j_p j_{p+1}\cdots j_{p+m}}$ is a tensor. The harmonic integrals, or the corresponding differential forms, are classified as effective and ineffective. A p-fold harmonic form is effective if a certain associated $(p-2)$-form (obtained by covariant differentiation) is a null form. It is shown that there are exactly R_p-R_{p-2} linearly independent effective p-fold harmonic forms. The ineffective harmonic forms are expressible in terms of the effective forms of lower order $p-2$, $p-4$, $\cdots$, and of certain differential forms which are defined projectively in terms of the ambient space S_r. This classification of the harmonic integrals is in close connection with Lefschetz's classification of the p-cycles on V_m as effective and ineffective: the effective p-fold integrals have non-vanishing periods only on the R_p-R_{p-2} effective p-cycles. The properties of the period matrix of the effective integrals are studied and it is shown that this matrix is invariantively related not only to the system of hyperplane sections of the original projective model but also to the entire continuous system containing the hyperplane sections. We shall mention one more application. A p-cycle Γ may be homologous to a cycle lying on some $(p-k)$-dimensional subvariety of V_m ($k\leqq[\frac{1}{2}p]$). It is shown that in that case the periods of certain harmonic integrals relative to Γ must be equal to zero. If $p=2q$ and $k=q$ we have thus necessary conditions that Γ be an algebraic cycle. Unfortunately it is not yet known whether these conditions are also sufficient. It is an important question for possible applications to algebraic geometry.

The last chapter deals with applications to continuous groups. It is shown how a metric can be defined on any manifold which admits a continuous group of transformations into itself, provided the group satisfy certain general conditions (semi-simple, closed, transitive Lie group). The invariant integrals of the group are studied and it is shown that the harmonic integrals lead to group invariants which have been discussed by Cartan. Several sections deal with special groups: the unimodular group, the orthogonal group and the symmetric group.

O. Zariski (Baltimore, Md.).

Differential Geometry

Haupt, Otto. Über ebene Bogen mit vorgeschriebenen Ordnungssingularitäten. Jber. Deutsch. Math. Verein. **50**, 256–269 (1940).

An arc $\mathfrak{B}$ in the Euclidean plane has the order n if the maximum number of its points of intersection with a straight line is n. The order of a point P of $\mathfrak{B}$ is defined as the order of a sufficiently small partial arc containing P in its interior. For all P of $\mathfrak{B}$ the tangent of P shall exist as the limit of all the secants through any two points of $\mathfrak{B}$ converging to P. The author proves for every $m\geqq 2$ the existence of arcs $\mathfrak{B}$ of order $2m+1$ such that the set $\mathfrak{P}$ of the points of order $2m+1$ on $\mathfrak{B}$ is perfect and its measure differs arbitrarily little from the length of $\mathfrak{B}$. He constructs $\mathfrak{B}$ as the Cartesian image of a strictly monotone and continuously differentiable function $y=y(x)$ $[0\leqq x\leqq 1]$ with the property that the set $\mathfrak{P}_x$ of the projections of $\mathfrak{P}$ on the x-axis is identical with any given perfect and nowhere dense set of sufficiently great

measure in the interval $0 \leqq x \leqq 1$ (the set of the zeros of dy/dx is also identical with $\mathfrak{P}_x$). *P. Scherk.*

Kasner, Edward and De Cicco, John. Families of curves conformally equivalent to circles. Trans. Amer. Math. Soc. **49**, 378–391 (1941). [MF 4338]

A study of the geometry of three-parameter families of curves in a plane conformally equivalent to all the circles of the plane. Two sets of characteristic properties are found. The two sets have a quasi-dual relationship to one another. The three properties of each set are directly or indirectly expressed in terms of families of tangent osculating parabolas. Minimal coordinates are used in carrying out the analysis. *P. Franklin* (Cambridge, Mass.).

Feld, J. M. Differential and integral invariants of plane curves and Horn angles. Bull. Amer. Math. Soc. **47**, 318–327 (1941). [MF 4185]

Defining a curve in the complex z-plane by a self-conjugate equation $f(z, \bar{z})=0$, the author obtains conformal invariants of horn angles as combinations of derivatives of z with respect to $\bar{z}$ by simple manipulations of these derivatives. Some of these invariants have already been obtained by other methods, but others are new. Similarly, an infinite set of invariants of a single curve under inversive transformations is obtained, and from these the author gets inversive invariants of horn angles of arbitrarily high orders.

R. J. Walker (Princeton, N. J.).

de Mira Fernandes, A. Axiomatics of spaces of lineal elements. Portugaliae Math. **2**, 7–12 (1940). (Portuguese) [MF 4280]

E. Cartan, in "Les espaces de Finsler" [Actual. Sci. Ind. **79**, Paris, 1934], has given five conditions for an n-dimensional space with a metric $g_{ik}(x, x')dx^i dx^k$, in which the g_{ik} depend on the x^i and their derivatives x'^i, and with a Euclidean connection. In this paper these five conditions are reduced to three with the aid of a theorem of the author [Atti Accad. Naz. Lincei. Rend. (6) **21**, 555–562 (1935)].

D. J. Struik (Cambridge, Mass.).

Hazebroek, P. Un problème de la théorie des réseaux plans. Nederl. Akad. Wetensch., Proc. **43**, 1172–1179 (1940). [MF 3727]

Let there be given a net of curves in a plane, not composed entirely of straight lines. The lines separating harmonically at each point of the plane the tangents to the curves of the given net are themselves tangent to the curves of a second net. The author proves that this second net is composed entirely of straight lines if and only if the given net is formed by a family of conics tangent to a fixed line, each conic of the family being tangent to every other conic of the family at a fixed point not on the fixed line. Moreover, this net of conics is the projection on a plane of the asymptotic net on a certain surface. Hence the surface sustains conjugate nets whose curves lie on planes passing through a fixed point. The method and notation used is that employed by E. Cartan. *V. G. Grove* (East Lansing, Mich.).

Foster, Malcolm. Note on autopolar curves. Bull. Amer. Math. Soc. **47**, 247–253 (1941). [MF 4168]

The author studies a particular contact-transformation, namely, the polarity with respect to the parabola $2y=x^2$. He studies the corresponding transformation for a differential equation $f(x, y, y', y'', \cdots, y^{(n)})=0$ and also the curves $y=f(x)$ which are autopolar, that is, are transformed into themselves by the said polarity. In this case $f(x)$ satisfies the equation

$$f[f'(x)]+f(x)-xf'(x)=0.$$

Let (x_1, y_1) and (x_2, y_2) be a pair of conjugate points on the curve $y=f(x)$, so that the polar of (x_1, y_1) is tangent to the curve at the point (x_2, y_2). The author shows that $y_1+y_2=x_1x_2$; furthermore $x_1=f'(x_2)$ and $x_2=f'(x_1)$. He studies the self conjugate points, and gives some geometrical properties of these points and of the conjugate pairs of points. He remarks that the equation $\phi(x)y''+\phi(y')=0$ is transformed into itself by the considered polarity, and determines in some cases whether there is some autopolar curve satisfying the preceding differential equation. *G. Fubini.*

Lilley, S. The characteristic exponents of a pair of power series. Proc. Cambridge Philos. Soc. **37**, 105–108 (1941). [MF 3770]

The characterization of the singularity at the origin 0 of a curve branch with the equations

$$(1) \qquad x=t^\rho, \quad y=t^\sigma \sum \beta_i t^{\mu_i}, \qquad \mu_0=0,$$

has been completed through the work of several mathematicians. The author considers the more general form

$$(2) \qquad x=t^\rho \sum a_i t^{\mu_i}, \quad y=t^\sigma \sum b_i t^{\mu_i},$$

wherein $\rho>\sigma$, $\mu_0=0$, $\mu_i<\mu_{i+1}$, $a_0=b_0=1$, as the equations of a plane curve branch. He characterizes the nature of the singularity at 0 by reducing (2) to the form (1) by a transformation of the parameter t, analytic at $t=0$. The only previous consideration of this more general form appears to be that of a special case by Coolidge [A Treatise on Algebraic Plane Curves, Oxford, 1931, pp. 221–222].

T. R. Hollcroft (Aurora, N. Y.).

Green, Louis. Twisted cubics associated with a space curve. II. Amer. J. Math. **63**, 352–360 (1941). [MF 4156]

[The first part appeared in the same J. **62**, 285–306 (1940); cf. these Rev. **1**, 268.] Referred to the tetrahedron of Halphen the nonhomogeneous point equations of a curve Γ have the form

$$y=x^2+\sum_7^\infty p_n x^n, \quad z=x^3+\sum_6^\infty q_n x^n,$$

and referred to the dual tetrahedron the nonhomogeneous plane equations of Γ are

$$\eta=\zeta^2+\sum_7^\infty \pi_n \xi^n, \quad \zeta=\xi^3+\sum_6^\infty \kappa_n \xi^n.$$

The coefficients $p_n=\pi_n$, $q_n=\kappa_n$ for every n if, and only if, Γ is an harmonic curve. In the first part of this paper the existence of a certain "self-dual" quadric surface Q is established. It is shown that $p_n=\pi_n$, $q_n=\kappa_n$ for every n if the "dual relative to Γ" of any point whatever is the polar of the point with respect to the Q. The one parameter family T_α of five-point cubics studied in I form a subset of a two-parameter family $T_{\alpha\beta}$ of five-point cubics. For a given value of β, these cubics have the same osculating conic, lie on the same quadric cone and determine the same linear complex. The principal point, line and plane of the pair Γ and $T_{\alpha\beta}$ are discussed. Two cubics of the family $T_{\alpha\beta}$ are called associate if they possess the same principal line. A cubic $T_{\alpha\beta}$ may be self-associate. A cubic $T_{\alpha\beta}$ which is not self-associate is called singular if it has (with Γ) the same principal point as its associate. These special pairs and types of the family $T_{\alpha\beta}$ are discussed quite exhaustively. *V. G. Grove.*

Scherk, Peter. On real closed curves of order $n+1$ in projective n-space. Proc. Nat. Acad. Sci. U.S.A. **27**, 181–182 (1941). [MF 3900]

Let C be a curve in real projective n-space for which at every point P osculating $L_\nu(P)$, $1\leqq\nu\leqq n-1$, exist. That means: if Q approaches P on C, the straight line PQ converges to $L_1(P)$, and if $L_{\nu-1}(P)$ is already defined, then $L_\nu(P)$ is the limit of the L_ν through $L_{\nu-1}(P)$ and Q. If $L_{n-1}(P)$ has only a finite number of points common with C, numbers $a_0, \cdots, a_{n-1}$, $a_i=1, 2$, exist such that any L_{n-1} containing $L_\nu(P)$ but not $L_{\nu+1}(P)$ crosses C at P if and only if $a_0+a_1+\cdots+a_\nu$ is odd. All the a_i, except at most one, equal 1. If all equal 1 the point P is called regular; if $a_\nu=2$, P is counted as an $(n-p)$-fold singular point. If C is a closed curve of real order $n+1$ then the sum of the multiplicities of its singular points is $\leqq n+1$ and $\equiv n+1 \pmod 2$.

H. Busemann (Chicago, Ill.).

v. Sz. Nagy, Gyula. Über die Kurven n-ter Ordnung im projektiven q-dimensionalen Raum für $n<2q$. J. Reine Angew. Math. **183**, 1–8 (1940). [MF 4224]

A set $K_n^{(q)}$ of closed curves (each of them may have multiple points) in real projective q-space has the order n if the maximum number of its points of intersection with the hyperplanes of the space is n. Obviously $n\geqq q$. If $n\geqq 2q$, it is well known for $q=2$ [cf. Juel, Danske Vid. Selsk. Skr. (7) **11**, no. 2, 113 (1914)] that $K_n^{(q)}$ may consist of any number of curves and have at the same time any number of double points. However, if $n<2q$, the total number of the curves and double points of $K_n^{(q)}$ is not greater than $n+1-q$ for all q. In this theorem, an m-fold point may be counted as $m-1$ double points. The proof is conducted by cutting $K_n^{(q)}$ with a conveniently chosen hyperplane. Furthermore, the author obtains generalizations and applications of this result and discusses special cases. *P. Scherk.*

Su, Buchin. Projective differential geometry of singularities of plane curves. J. Chinese Math. Soc. **2**, 139–151 (1940). [MF 4202]

Let C be a plane curve which has a singular point P of order m. The author shows that, if P has a number of special properties, it is possible to determine a unique triangle in the neighborhood of order $2m$ at P which is projectively related to the curve. *A. Fialkow.*

Su, Buchin. Contributions to the projective theory of curves in space of n dimensions. I. J. Chinese Math. Soc. **2**, 153–173 (1940). [MF 4203]

A number of theorems concerning the plane sections of the circumscribed developable of a curve in a projective n-dimensional space S_n ($n\geqq 3$) are established. It is assumed that the planes which form the sections are in the osculating space S_3 of the curve at one of its points P. In particular, it is shown that the plane section made by a plane π (distinct from the osculating plane) which passes through the tangent of the curve at P has an inflexion at P. Consequently it admits an invariant point O_6, namely, the cusp of the cubic which has contact of order 6 with the plane section at P. As the plane π turns about the tangent the corresponding point O_6 describes a twisted cubic in the osculating space S_3 of the curve at P. *A. Fialkow.*

Su, Buchin. Contributions to the projective theory of curves in space of n dimensions. II. J. Chinese Math. Soc. **2**, 277–289 (1940). [MF 4213]

Various generalizations of the theorem quoted in the preceding paper are considered. The author assumes that the plane π which makes the section on the circumscribed developable of the curve passes through the tangent to the curve at one of its points P and is contained in the osculating space S_m ($3\leqq m\leqq n$) at P but not in the osculating space S_{m-1}. *A. Fialkow* (Brooklyn, N. Y.).

Su, Buchin. Some arithmetical invariants of a curve in projective space of n dimensions. Univ. Nac. Tucumán. Revista A. **1**, 143–157 (1940). [MF 4060]

Two theorems concerning a curve Γ in an n-dimensional projective space S_n are derived, each of which yields a set of arithmetic invariants for the curve at its regular points. The proofs depend upon the calculation of some Wronskians which are associated with Γ. The following theorem illustrates the type of results obtained: Given a curve Γ in S_n and its regular point P, consider the hypercone projecting the osculating spaces S_{n-3} of Γ from P and the developable hypersurface circumscribed to Γ. If a plane α be contained in the osculating space $S_m(P)$ of Γ at P ($3\leqq m\leqq n$) and intersects the tangent $S_1(P)$ of Γ, then the sections (C_α) and (Γ_α) of the hypercone and the developable hypersurface made by α are tangent to each other, and the invariant of Segre of (Γ_α) with respect to (C_α) is equal to

$$m(m-1)^{-(m-1)/(m-2)},$$

and is therefore independent of α, P and the curve Γ. If $m=3$, this result coincides with a previous theorem of the author [see the preceding review]. *A. Fialkow.*

Fon, Te-Chih. Some new geometrical significances of the projective curvatures and the curvature form of a space curve. J. Chinese Math. Soc. **2**, 193–197 (1940). [MF 4205]

The first and second projective curvatures I, J and the curvature form $Id\sigma^2$ (σ = projective arc) of a space curve C at a point P are given geometric significance by expressing them in terms of double ratios of elements determined by certain geometrical configurations projectively associated with C at P. The configurations employed are the normal tetrahedron of B. Su, the point of Sannia and the osculating conic. *J. L. Vanderslice* (Bethlehem, Pa.).

Hsiung, Chuan-Chih. On the curvature form and the projective curvatures of curves in space of four dimensions. Univ. Nac. Tucumán. Revista A. **1**, 159–171 (1940). [MF 4061]

The purpose of this paper is to express certain invariants of curves in a space of four dimensions found by B. Su [Sci. Rep. Univ. Chekiang **2**, 115–169 (1937)] in terms of double ratios of covariant points. *V. G. Grove.*

Hsiung, Chuan-Chih. Note on the intersection of two space curves. Tôhoku Math. J. **47**, 201–209 (1940). [MF 4021]

Bompiani [Mem. Accad. Sci. Ist. Bologna (8) **3**, 35–38 (1926); Atti Accad. Naz. Lincei. Rend. (6) **14**, 456–461 (1932)] has shown the existence of certain principal lines and points associated with two space curves intersecting as a point O. The author gives other geometrical interpretations for these lines and points. The methods used are those employed by Bompiani in the papers cited. In particular the following characterization of the principal lines is made: Let the space curves C, $\bar{C}$ intersect at O in such a way that the tangent lines t, $\bar{t}$ and the osculating planes τ, $\bar{\tau}$ at O do not coincide. Let π be the plane $(t, \bar{t})$ and p the line $(\tau, \bar{\tau})$,

and assume that p coincides with neither t nor $\bar{t}$. Let T, $\bar{T}$ denote the intersection of t, $\bar{t}$ with the quadrics Q, $\bar{Q}$ having three-point contact with C, $\bar{C}$, respectively. Let q denote the intersection of the tangent planes to Q, $\bar{Q}$ at O; r denote the intersection of π with the plane (pq), and let R be the intersection of T, $\bar{T}$ with the harmonic conjugate of r with respect to t, $\bar{t}$. If the condition be imposed on Q, $\bar{Q}$ that they pass through R, then π intersects Q, $\bar{Q}$ in conics whose intersections other than O, R determine with O the principal lines. The principal points are characterized as follows: Let C^*, $\bar{C}^*$ be the projections of C, $\bar{C}$ from any point on t, $\bar{t}$ on τ, $\bar{\tau}$, respectively, and let P, $\bar{P}$ be the poles of p with respect to any four point conics of C, $\bar{C}$ at O. Then the line $P\bar{P}$ intersects the principal lines in the principal points. A discussion is also made of the case in which τ and $\bar{\tau}$ coincide. *V. G. Grove* (East Lansing, Mich.).

Hsiung, Chuan-Chih. On the plane sections of the tangent surface of a space curve. J. Chinese Math. Soc. **2**, 239–245 (1940). [MF 4211]

This paper concerns itself with the study of the sections of the tangent developable surface of a twisted curve through the tangent line of the curve. The problem was previously studied by B. Su [Tôhoku Math. J. **38**, 233–244 (1933), and J. Chinese Math. Soc. **2**, 98–137 (1937)], making use of certain osculants introduced by Bompiani [Boll. Un. Mat. Ital. **5**, 118–126 (1926)]. It is found among other things that among the plane sections of the tangent surface there exists one and only one whose seven-point cuspidal cubic also has eight-point contact; there are only two sections for which the eight-point cubic through the cusp of the seven-point cuspidal cubic has nine-point contact with the plane section; as the sectioning plane revolves about the tangent line the locus of the cusp of the seven-point cuspidal cubic is a twisted cubic; the locus of the line of cusps of the six-point cuspidal cubics is a quadric cone; the locus of the cuspidal tangent is a cubic ruled surface having the tangent to the space curve as a triple line; the locus of the seven-point cuspidal cubic is a surface of order six.

V. G. Grove (East Lansing, Mich.).

Srinivasiengar, C. N. Some properties of rectilinear congruences. Proc. Indian Acad. Sci., Sect. A. **12**, 350–353 (1940). [MF 3762]

The purpose of this paper is to generalize certain well-known properties of ruled surfaces associated with a line of a congruence. These generalizations may be illustrated by the following theorems: There are two ruled surfaces through a given line of the congruence having their central points at a given point on that line. The sum of the parameters of distribution for those two surfaces is a constant as the point generates the line. The central planes of these surfaces make complementary angles with the central plane of either mean surface. Let A and B be two distinct points equally distant from the midpoint between the foci on the line. Then the parameters of distribution for the ruled surfaces having their central points at A are equal to those having their central points at B. *V. G. Grove.*

Bottema, O. Strahlenkongruenzen mit einem zweigliedrigen System von quadratischen Regelscharen. Nederl. Akad. Wetensch., Proc. **43**, 1276–1281 (1940). [MF 3940]

This paper gives the proof of the theorem: The only line congruences in which ∞^2 reguli are contained are (a) the congruence (1, 3) of the secants of a cubic of R_3 and the dual congruence; (b) a special congruence (2, 2), consisting of lines which are tangent to a quadric surface and meet a straight line tangent to the same surface; (c) the congruence (1, 2) of lines meeting a conic and a line intersecting the conic; (d) the linear congruences (1, 1). The author uses the mapping of the straight lines of R_3 on a four-dimensional manifold of second order in R_5. Each regulus corresponds to a conic on this manifold. The proof of the given theorem is based on a theorem due to Darboux and Segre that a two-dimensional manifold containing ∞^2 conics is either Veronese's surface of the fourth order in R_5, or a projection of it. *E. Helly* (Paterson, N. J.).

Takeda, K. On line congruences. V. Tôhoku Math. J. **47**, 181–187 (1940). [MF 4019]

This paper is a continuation of papers previously reviewed [cf. these Rev. **1**, 170 and **2**, 17]. *V. G. Grove.*

Sintsov, D. Théorie générale du connexe aux éléments (point, droite) dans l'espace. Acad. Sci. RSS Ukraine. Rec. Trav. [Zbirnik Prace] Inst. Math. **1940**, no. 5, 31–71 (1940). (Ukrainian. Russian and French summaries) [MF 3750]

There are ∞^7 point-line elements in R_3, each consisting of an arbitrary point x and an arbitrary line p. A point-line connex is the ∞^6-manifold of point-line elements, which is defined by one algebraic equation $f(x, p)=0$, where x and p stand for the coordinates of the point x and the line p, respectively. A single, double, triple, $\cdots$ coincidence is the manifold of all point-line elements belonging to two, three, four, $\cdots$ connexes. The author gives formulas for the characteristic numbers of coincidences, if the degrees of the defining equations with respect to the point and line coordinates are known. The polar-connex with respect to a point or a line and the higher polars are used, as with algebraic surfaces, to investigate singular manifolds of point-line elements and of points and lines, which are connected with a given connex or a given coincidence. Another chapter is devoted to the study of the "principal" coincidence of a given connex or coincidence; this is the set of all point-line elements of the given manifold such that the point of the element is on the corresponding line. The connection of these principal coincidences with problems of integration is mentioned. *E. Helly* (Paterson, N. J.).

Su, Buchin. A note on the planar point of a surface. Univ. Nac. Tucumán. Revista A. **1**, 95–103 (1940). [MF 4057]

A planar point O of a surface is one at which every plane section has an inflection at O. At such a point there are three triple point tangents. The author supplements previous investigations by V. G. Grove and I. Popa on planar points by determining certain covariant points, lines and planes associated with the three triple point tangents at O and exhibiting some of their properties.

J. L. Vanderslice (Bethlehem, Pa.).

Durañona y Vedia, A. A proof of the Gauss Bonnet theorem. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie 2: Revista **4**, 325–327 (1940). (Spanish) [MF 4200]

Terracini, Alejandro. On the existence of surfaces with prescribed principal lines. Union Mat. Argentina, Publ. no. **16**, 22 pp. (1940). (Spanish) [MF 3935]

Properties of the principal curves of surfaces in a projective space of five dimensions were first given by C. Segre [Rend. Circ. Mat. Palermo **30**, 87–121 and 346–348 (1910)]

and further developed by E. Bompiani [Atti Accad. Sci. Torino **48**, 393–410 (1913)] and by A. Terracini [Scritti mat. offerti a Luigi Berzolari, Pavia, 1936, pp. 449–478]. A characteristic feature of the surface of Veronese from this standpoint was shown by C. Segre [Atti Accad. Naz. Lincei. Rend. (5) **30**, 200–203 and 227–231 (1921)]. Systems of 5-webs of plane curves, as developed by W. Blaschke [Abh. Math. Sem. Hansischen Univ. **9**, 313–317 (1933)], are included among the systems of principal curves of surfaces in S_5. Fundamental questions of existence, still unanswered, are pointed out by Blaschke and G. Bol [Geometrie der Gewebe, Berlin, 1938]. A first step in this direction was made by A. Terracini [C. R. Acad. Sci. Paris **208**, 616–618 (1939)]. The cross-ratio of four points on the u-curve and its analogue on the v-curve each reduces to the same fundamental differential form in the limiting case. This form is the first member of the differential equation of the systems of principal curves. This result is a generalization of one previously found by E. Bompiani for surfaces in S_3 [Boll. Un. Mat. Ital. **5**, 167–173 and 209–214 (1926)]. Two questions are proposed in the present paper: (A) Given arbitrarily a differential equation, with unknown functions for coefficients, does a surface exist such that this equation defines its principal curves? (B) Given arbitrarily the differential fractional form, does a surface exist such that its fundamental differential form coincides with it? Both questions are answered affirmatively. *V. Snyder.*

Terracini, Alejandro. Surfaces with parabolic projections. Univ. Nac. Tucumán. Revista A. **1**, 221–254 (1940). (Spanish) [MF 4065]

This paper has for its purpose the determination of surfaces in S_5 which project into surfaces of S_4 of parabolic type. It is not proposed to make a systematic study or complete enumeration of such surfaces, but rather to provide some examples that satisfy the required conditions. It is based on the paper reviewed above, which includes the definitions and references to related papers. The principal result is that the curves of a surface S which become characteristic curves on the projection surface of parabolic type constitute a system of principal curves on the given surface.

The centers of projection lie on the given surface, and are called centers of parabolic projection. It is shown that not more than five such centers exist, except for the surface of Veronese, which contains an infinite number. From the conditions of integrability of the differential equation of the principal curves of the given surface, the conditions for a parabolic lattice on the projection surface, as defined by B. Segre [Ann. Fac. Sci. Univ. Toulouse (3) **20**, 1–46 (1928); see nos. 25, 45], follow. If a system of principal curves on S of S_5 is known to be a parabolic lattice, and is projected from a line r into the asymptotic system of a surface in S_3, this system is necessarily formed by virtual characteristics. After a detailed analysis of the conditions which insure one parabolic center on the given surface, the presence of two is considered. This system includes, as projections in S_3, the W-surfaces of Klein and ruled surfaces having a rectilinear directrix. Then follows the more complicated case of surfaces with three parabolic centers, and one instance of a surface with four. *V. Snyder* (Ithaca, N. Y.).

Rasmusen, R. B. and Hagen, B. L. Comments on canonical lines. Bull. Amer. Math. Soc. **47**, 298–302 (1941). [MF 4181]

This note deals with the projective geometry of surfaces referred to a conjugate net, results being expressed in terms of the invariants of the net. The equations of the canonical edges of Green are derived using a result of Segre and necessary and sufficient conditions given that the first (second) edge of Green belong to the first (second) canonical pencil of W. M. Davis. A two parameter family of quadrics associated with each point of the net is introduced as a generalization of the quadric of Davis.

J. L. Vanderslice (Bethlehem, Pa.).

Finikoff, S. Couple de surfaces en correspondance biunivoque dont les axes homologues relativement à la base de la correspondance coïncident. Univ. Nac. Tucumán. Revista A. **1**, 313–331 (1940). [MF 4071]

Let there be given a surface (M) generated by a point M, and a conjugate net B on (M). Let M_1, M_2 be the Laplace transforms of M with respect to B. The osculating planes at M to the curves of B intersect in a line called the (first) axis. The line M_1M_2 is called the ray (second axis) of B at M. Let (N) be a surface generated by a point N and in one-to-one correspondence with (M), so that M and N are corresponding points and such that the net B' on (N) corresponding to B is also conjugate. The author considers the two problems: (A) Is it possible for the axes of B and B' and simultaneously for the rays of B and B' to coincide? (B) Is it possible for the axis of B to coincide with the ray of B' and at the same time for the ray of B to coincide with the axis of B'?

It is shown that the nets B, B' in problem (A) must be nets of Wilczynski, that is, each of the congruences of tangents to the curves of the net belongs to a linear complex [E. J. Wilczynski, Acad. Roy. Belgique. Cl. Sci. Mém. (2) **3** (1911)]. If B is a net of Wilczynski there exists a one parameter family of transversal surfaces Σ of the axis congruence of B whose tangent planes at their points of intersection with the axis at M pass through the ray of B, and a one parameter family of transversal surfaces Σ' of the ray congruence of B whose tangent planes pass through the axis of B. These surfaces may be grouped into two families (Σ) and (Σ') such that two surfaces of (Σ') will serve as the focal surfaces of the congruences of tangents to the conjugate nets on a pair of surfaces of (Σ), and reciprocally that pairs of (Σ) will serve as focal surfaces for the congruences of the nets on the surfaces Σ' of (Σ'). Two such surfaces Σ (or Σ') form a solution of the problem (A).

On the other hand a surface Σ and a surface Σ' satisfy the conditions of problem (B), but this solution is not the most general one for that problem. It is shown, however, that, if the tangents to nets R belong to a linear complex and are related by a transformation of Jonas, then the axis of each net coincides with the ray of the other. And moreover this is the only other non-trivial solution of problem (B).

The method and notation is that of E. Cartan. The coordinates M_0^i, M_1^i, M_2^i, M_3^i of the points M, M_1, M_2, M_3 satisfy differential equations of the form $dM_i=w_i^kM_k$, $i, k=1, 2, 3, 4$, wherein $w_i^k=a_i^kdu+b_i^kdv$, and wherein the functions a_i^k, b_i^k are subject to certain integrability conditions. *V. G. Grove* (East Lansing, Mich.).

Tompkins, C. A flat Klein bottle isometrically embedded in Euclidean 4-space. Bull. Amer. Math. Soc. **47**, 508 (1941). [MF 4547]

Frucht, Roberto. Zur Geometrie auf einer Fläche mit indefiniter Metrik. Union Mat. Argentina, Publ. no. **11**, 22 pp. (1940). (German. Spanish translation) [MF 3851]

A surface with an indefinite first fundamental form (that

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is, $EG-F^2<0$) determines a real surface Σ with negative curvature in affine differential geometry whose first affine fundamental form has coefficients $E, 2F, G$. Take the asymptotic lines on Σ as parameter lines and determine for a given point (u_0, v_0) the values h, k, h_1, k_1 such that the three quadrangles $[(u_0-h, v_0-k), (u_0, v_0-k), (u_0-h, v_0), (u_0, v_0)]$, $[(u_0, v_0-k), (u_0+h_1, v_0-k), (u_0+h_1, v_0), (u_0, v_0)]$ and $[(u_0-h, v_0), (u_0-h, v_0+k_1), (u_0, v_0+k_1), (u_0, v_0)]$ have equal affine area w. Designate by w' the area of $[(u_0, v_0), (u_0+h_1, v_0), (u_0, v_0+k_1), (u_0+h_1, v_0+k_1)]$. Then $\lim (w-w')/w^2$ equals the Gauss curvature of the corresponding point of the original surface. This result yields also intuitive interpretations of the geodetic curvature of curves on the original surface.

H. Busemann (Chicago, Ill.).

Alexandroff, A. Existence of a given polyhedron and of a convex surface with a given metric. C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 103–106 (1941). [MF 4261]

Given a complex C of plane triangles, homeomorphic to the surface of the sphere, and of sides and angles of given sizes, such, however, that the sum of angles at no vertex exceeds 2π; does there exist a convex polyhedron in the Euclidean 3-space for whose surface the given complex presents a (congruent) triangulation? The author answers this problem in the affirmative and proposes a proof based (a) on an induction with respect to the number of vertices, (b) on a continuous change of the metrization of C in order to continuously produce the totality of polyhedral solutions for all possible C. Not all the essential details of this proof are published.

H. Lewy (Berkeley, Calif.).

Caccioppoli, Renato. Ovaloidi di metrica assegnata. Pont. Acad. Sci. Comment. **4**, 1–20 (1940). [MF 4108]

The paper is devoted to a proof of the following theorem: "To every definite positive quadratic differential form (given on a sphere) which is sufficiently regular and has positive curvature there corresponds an ovaloid uniquely determined except for its position in space, which admits the given form as the square of its linear element." The situation is interpreted in terms of a transformation between two function spaces, the points of one of these spaces being ovaloids (that is, closed convex surfaces in ordinary space) given by their coordinates as functions defined on the surface of a sphere, while the points of the other are "abstract ovaloids" given by differential forms whose coefficients are also functions defined on the surface of the sphere. The question reduces then to the question of the existence and uniqueness of the inverse of the above transformation. The discussion follows a general procedure of dealing with inverses of transformation between two function spaces which the author has treated in a series of notes in the Rendiconti of the Accademia Nazionale dei Lincei between 1932 and 1936. This procedure consists of two parts: (1) proof that the differential of the transformation possesses an inverse, which gives local invertibility, and (2) the passage to invertibility in the large, which requires a proof that compact sets of the second space are images of compact sets of the first. The principal difficulties are encountered in the first part, which is based on a discussion given by H. Weyl [Vierteljschr. Naturforsch. Ges. Zürich **61**, 40–72 (1916)]. The transformation obtained first falls short of being the inverse in that the functions furnished by it do not possess the required degree of regularity (as expressed in terms of existence, etc., of partial derivatives of certain orders); the difficulty is overcome by restricting the consideration in a part of the paper to conformal mappings which, however, does not affect the generality of the final result.

G. Y. Rainich (Ann Arbor, Mich.).

Levi-Civita, T. Formula di Green e di Stokes. Univ. Nac. Tucumán. Revista A. **1**, 23–33 (1940). [MF 4053]

The formulas of Green and of Stokes can be derived by considerations which do not involve any postulates of metrical geometry either of Euclidean or Riemannian space. The formulas with and without metrical specifications receive particular attention for the case of a Riemannian manifold of two and three dimensions.

D. J. Struik.

Tolotti, Carlo. Sul calcolo dei simboli di Riemann per ds^2 assegnati qualsiasi. Pont. Acad. Sci. Acta **3**, 35–46 (1939). [MF 4095]

Let $\sum a_{ik}dx^i dx^k$ be a quadratic differential form with non-vanishing determinant and denote by $d, \delta, d', \cdots$ symbols of differentiation. Put

$$\tau^h(d, \delta) = \delta dx^h + \sum \Gamma^h_{ij} dx^i \delta x^j.$$

According to Levi-Civita the second differentials $d\delta, dd', \cdots$ are defined by $\tau^h(d, \delta)=0$, $\tau^h(d, d')=0, \cdots$. A differential of third order is defined by supposing that

$$d'd\delta x^h = -d'(\sum \Gamma^h_{ij} dx^i \delta x^j)$$

(because $d\delta x^h = -\sum T^h_{ij} dx^i \delta x^j$) and by calculating $d'dx^i$ and $d'\delta x^j$ according to the preceding definition of second differentials. In this manner Levi-Civita proved that

$$(d'\delta' - \delta' d')dx^h = \sum R^{(h)}_{i,kl} x^i d'x^k \delta' x^l.$$

This equation determines the Riemann symbols of second kind; it can be written in the equivalent form

$$(*) \quad \delta' \sum \Gamma^h_{kl} dx^k d'x^l - d' \sum \Gamma^h_{kl} dx^k \delta' x^l = \sum R^h_{i,kl} dx^i d'x^k \delta' x^l.$$

This equation contains no differentials of third order, and therefore remains true even if we change the definition of these differentials; we may for instance, following Pérès, suppose that $\delta' d' dx^h = d' d\delta' x^h = d\delta' d' x^h$. In this case (*) is equivalent to

$$\delta' \tau_j(d, d') - d' \tau_j(d, \delta') = \sum_{i,k,l} R_{ij,kl} dx^i d'x^k \delta' x^l,$$

where $R_{ij,kl}$ are the Riemann symbols of first kind. The author compares this formula with another identity due to Riemann, and shows that it leads to a computation of $R_{ij,kl}$ which is easier than the direct computation by using the definition of these symbols.

G. Fubini.

Wong, Yung-Chow. Generalized helices in an ordinary V_n. Proc. Cambridge Philos. Soc. **37**, 14–28 (1941). [MF 3767]

A linear vector m-space R_m, $m<n$, defined along a curve C in a Riemannian space of n dimensions and lying in the largest osculating vector space of C (dimension h) is called characteristic if it is autoparallel along C and makes constant angles with its tangent and principal normals. The case $m=1$ has been studied by H. A. Hayden [Proc. London Math. Soc. (2) **32**, 337–345 (1931)] and some types of the case $m=2$ by M. Syptak [C. R. Acad. Sci. Paris **198**, 1665–1667 (1934)]. This paper establishes a general condition for the case $m=2$. If the characteristic R_2 is not necessarily orthogonal to any of the principal directions its curvatures are proportional to a set of constants, and the number of

such R_2 is $\frac{1}{2}h$ if h is even and $\frac{1}{2}(h-1)$ when h is odd. Some theorems on the case of general m conclude the paper.

D. J. Struik (Cambridge, Mass.).

Schwartz, Abraham. The Gauss-Codazzi-Ricci equations in Riemannian manifolds. J. Math. Phys. Mass. Inst. Tech. **20**, 30–79 (1941). [MF 3842]

The author considers in great detail the dependences and interrelationships existing between the Gauss-Codazzi-Ricci equations for a Riemannian space V_m in a V_n. The work makes essential use of the normal spaces defined by Schouten and Struik [Einführung in die neueren Methoden der Differentialgeometrie] and W. Mayer [Trans. Amer. Math. Soc. **38**, 267–309 (1935)]. Among the many results obtained the author emphasizes one which contains a theorem of the reviewer as a special case: for a Riemann space V_{n-1} embedded in a Euclidean space of n dimensions the Codazzi equations are consequences of the Gauss equations if the rank of the second fundamental form of V_{n-1} is not less than 4. [Cf. C. B. Allendoerfer, Amer. J. Math. **61**, 633–644 (1939); these Rev. **1**, 28.] *T. Y. Thomas.*

Haimovici, M. M. Sur la géométrie intrinsèque des surfaces non holonomes V_3^2. Univ. Nac. Tucumán. Revista A. **1**, 173–188 (1940). [MF 4062]

The geometry of a non-holonomic surface V_3^2 imbedded in a 3-dimensional Riemann space is developed by using the results of E. Cartan on the equivalence of Pfaffian systems. It is shown that the problem reduces to the study of three Pfaffian equations. The normal to the surface, torsion and curvature of the surface are defined in an intrinsic manner. If the torsion is not identically zero, an equivalence theorem for non-holonomic surfaces is proved which does not depend upon the curvature. If the torsion vanishes identically, the metric defined on the V_3^2 may also be considered as the metric of a 2-dimensional Riemann space whose total curvature is equal to the curvature of the V_3^2. As an application of the results, the defining Pfaffian equation and the metric of the non-holonomic surfaces of zero torsion and constant curvature are obtained in explicit form. *A. Fialkow.*

Thomas, T. Y. The characterization of flat Riemann spaces as a boundary value problem. J. Math. Phys. Mass. Inst. Tech. **19**, 247–259 (1940). [MF 3837]

In relativity theory the equations obtained by requiring that the contracted curvature tensor shall vanish $B_{\alpha\beta}=0$ are in some sense analogous to the Laplace equation $\nabla^2\phi=0$. In the former equations the components of the metric tensor $g_{\alpha\beta}$ play the same role as the ϕ in the Laplace equation. Now the harmonic function ϕ is completely determined within a convex domain when its values are given on the boundary of that domain. The question is raised by the author whether a similar theorem exists in connection with the equations $B_{\alpha\beta}=0$ and the metric tensor $g_{\alpha\beta}$ in a Riemann space R of class C^r and dimensionality $n>2$. Since the problem is to be attacked by means of n and $n-1$ dimensional volume integrals, the author first studies the various invariants connected with such integrals. With the aid of these invariants and Green's theorem the following elegant result is obtained: "let Σ be a hypersurface which satisfies the assumptions A and let h be a Riemann metric tensor defined on Σ and its interior I and satisfying the continuity and differentiability conditions B. Then there is one, and only one, set of quantities $h_{\alpha\beta}$ constituting the components of a tensor h, namely those given by $h_{\alpha\beta}=\delta_\beta^\alpha$ in $I+\Sigma$ such that (1) the coordinates are normal in the region I, (2) the system of differential equations $B_{\alpha\beta}=0$ is satisfied in I by the components $h_{\alpha\beta}$, and (3) the components $h_{\alpha\beta}$ assume the values δ_β^α on the boundary Σ." The author concludes with the further interesting result that if the equations $B_{\alpha\beta}=0$ are valid and if the volume of some sphere of radius r in the space R (class C^4) is equal to the volume of the Euclidean sphere of the same radius, then the curvature tensor of R vanishes in the interior of the sphere. [Reviewer's note. The equation (4.1) contains a typographical error; it should read $h_{\alpha\beta}(y)y^\beta=h_{\alpha\beta}(0)y^\beta$. The derivation of equation (4.3) is omitted by the author. This result is obtained by straightforward but considerable calculations involving normal coordinates.] *N. Coburn* (Austin, Tex.).

Yano, Kentaro. Concircular geometry III. Theory of curves. Proc. Imp. Acad. Tokyo **16**, 442–448 (1940). [MF 4324]

[The first two notes appeared in the same volume, pp. 195–200 and 354–360; cf. these Rev. **2**, 165.] A concircular transformation has previously been defined by the author as a particular type of conformal transformation of a Riemann space V_n, not necessarily on itself, which changes every geodesic circle of V_n into a geodesic circle. In this sense, concircular geometry is a generalization of inversive geometry. In this paper the author obtains a system of Frenet equations of a curve C in V_n which has an invariant meaning under concircular transformations of V_n. These equations involve a number of vectors (analogous to the successive normals of C in Riemannian geometry) whose directions remain unchanged by concircular transformations and a number of scalars (incompletely analogous to the curvatures of C) which are relative invariants with respect to concircular transformations. As an application, the author proves the theorem: If any geodesic circle of a hypersurface V_{n-1} can be regarded as a geodesic circle of the enveloping space V_n, then the hypersurface is totally umbilical and the mean curvature is constant on the hypersurface. The Frenet equations of this paper (after slight changes) are special cases of those obtained by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **26**, 437–439 (1940); these Rev. **2**, 21] when the transformations of V_n are any conformal transformations, not necessarily concircular. *A. Fialkow.*

Yano, Kentaro. Concircular geometry IV. Theory of subspaces. Proc. Imp. Acad. Tokyo **16**, 505–511 (1940). [MF 4325]

[Cf. the preceding review.] The concircular geometry of the subspaces of a Riemann space V_n is investigated. Necessary and sufficient conditions that a concircular transformation of V_n induce a like transformation on a subspace of V_n are found. In particular, it is shown that the transformation induced on a totally umbilical subspace by a concircular transformation of V_n is always concircular. A number of differential equations somewhat analogous (in a formal sense) to the classical Gauss-Codazzi equations are derived. The various tensors in these equations are not all concircular tensors; some of them only have a metric meaning. A number of known theorems are derived as simple consequences of the author's work. The results of this paper are related to the inversive theory of a surface in Euclidean 3-space developed by Blaschke and Thomsen [Vorlesungen über Differentialgeometrie, vol. 3, Berlin, 1928] and by Takasu [Differentialgeometrien in den Kugelräumen, vol. 1, Tokyo 1938] as well as to the conformal theory of the subspaces of V_n studied by the reviewer [Abstract 156, Bull. Amer. Math. Soc. **47**, 212 (1941)]. *A. Fialkow.*

Takasu, Tsurusaburo. Zur Theorie der Minimalkurven im kowformen Raume. Tôhoku Math. J. **47**, 129–158 (1940). [MF 4015]

By a uniform geometrical method based in great part on the first volume of his book "Differentialgeometrien in den Kugelräumen" [Tokyo, 1938] the author derives all the known forms of the natural equation of minimal curves in conformal 3-space in order to clarify their relations and exceptions. His results give rise at the same time to a theory of curves in Lie's higher circle geometry with the circles of curvature as elements. *J. L. Vanderslice.*

Takasu, Tsurusaburo. Laguerre-geometrische Verallgemeinerung eines Satzes von Carleman über die Isoperimetrie auf den Minimalflächen im weiteren Sinne. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1939**, no. 23, 14 pp. (1940). [MF 4000]

If R is a simply connected Jordan region in the plane with area a and perimeter l, then we have the isoperimetric inequality $a \leqq l^2/4\pi$. This inequality has been shown to hold on minimal surfaces [Carleman, Math. Z. **9**, 154–160 (1921)] and more generally on surfaces of negative Gauss curvature [Beckenbach and Radó, Trans. Amer. Math. Soc. **35**, 662–674 (1933)]. The purpose of the present paper is to extend this inequality to minimal surfaces in Laguerre geometry. The method is analogous to that used by Carleman.

T. Radó (Columbus, Ohio).

Kubota, Tadahiko. Einige Bemerkungen zur Takasuschen Arbeit über L-Minimalflächen. Tôhoku Math. J. **47**, 172–176 (1940). [MF 4017]

The author gives simplified proofs of two theorems of T. Takasu on L-minimal surfaces appearing in Sci. Rep. Tôhoku Imp. Univ., Ser. I. **22**, 1109–1154 (1933). He then generalizes the formula and theorem of Schwarz to minimal surfaces in Euclidean n-space. *J. L. Vanderslice.*

Hombu, Hitoshi. Grundlage der Geometrie in der Mannigfaltigkeit der Kurvenelemente. Proc. Imp. Acad. Tokyo **16**, 90–96 (1940). [MF 2221]

This paper summarizes previous results of the author and of the reviewer and adds a few new results. A transformation of the parameter $\bar{t}=\bar{t}(t)$ leads to

$$(*) \qquad x^{(r)\bar{i}} = \sum_{s=1}^{r} \alpha_s^r x^{(s)i},$$

where $x^{(s)\bar{i}} = d^s x^{\bar{i}}/d\bar{t}^s$ and $\partial \alpha_s^r/\partial \bar{t}^{(q)} = \binom{r}{q} \alpha_{s-q}^{r-q}$ with $1 \leqq q \leqq r$. It follows that two line elements $x^{(r)i}$ and $x^{(s)\bar{i}}$ belong to the same curve element if and only if (*) holds. Similarly the author states the conditions for a function f or a differential form $P(d)$ to belong to a curve element. Considering either $\bar{t}^{(a)}$ or $t^{(a)}$ and $d\bar{t}^{(a)}$ as parameters, one is led to two parameter groups $G_{(r)}$ and $h_{(r)}$, respectively. By a linear system is then meant a system of N quantities f^l or of N Pfaffians $P^l(a)$ and N' quantities $f^{l'}$ which are invariant under $G_{(r)}$ or $h_{(r)}$, respectively. The author investigates such linear systems. Some of the results have been published previously by the reviewer [Proc. Imp. Acad. Tokyo **13**, 224–237 (1937)]. *A. Kawaguchi* (Sapporo).

Berwald, L. On Finsler and Cartan geometries. III. Two-dimensional Finsler spaces with rectilinear extremals. Ann. of Math. (2) **42**, 84–112 (1941). [MF 3676]

The principal problem (due to P. Funk) solved in this paper is that of characterizing in an invariant manner all two-dimensional Finsler spaces which can be mapped geodesically on a Euclidean plane ("Finsler spaces with rectilinear extremals"). In case the Finsler space is Riemannian, this problem reduces to the classic one of Beltrami solved by surfaces of constant curvature. The introductory section presents a brief exposition of the general theory of two-dimensional Finsler spaces, as developed especially by the author and E. Cartan. Section II develops criteria for the geodesic equivalence of a two-dimensional Finsler space to a Euclidean plane. These criteria are obtained in the form of two conditions which involve respectively the "main scalar" I [Berwald] through certain of its geometric derivatives and the "curvature" K [A. Underhill] through its geometric derivatives. The separate meaning of each of the two conditions by itself is brought out by an alternative method of derivation. In section III all two-dimensional Finsler spaces with rectilinear extremals are determined whose main scalar I is a function of position only. It is shown that for such spaces I is always a constant, and so is K provided that $I \neq \pm 3/\sqrt{2}$. Under this proviso there are five types of spaces of the specified kind, each depending on a finite number of arbitrary constants. For the two exceptional values of I the corresponding Finsler space depends on an arbitrary function of one variable. Finally, all Landsberg spaces with rectilinear extremals are determined. A Landsberg space is one for which $I_s=0$, the subscript s denoting geometric differentiation along the extremal issuing from the lineal element (x, x') on which I depends.

J. Douglas (New York, N. Y.).

Finsler, Paul. Über eine Verallgemeinerung des Satzes von Meusnier. Vierteljschr. Naturforsch. Ges. Zürich **85** Beiblatt (Festschrift Rudolf Fueter), 155–164 (1940). [MF 4414]

The theorem of Meusnier for curves on m-dimensional manifolds in Euclidean n-space is generalized to curves on manifolds immersed in a Finsler space. To see how the usual definitions are to be modified it will suffice to give the definition of curvature. To define the angle between two directions at a point P take the points X, Y where the two directions intersect the indicatrix i of the space at P and let the hyperplane through Y parallel to the tangent plane of i at X intersect the direction PX at $\bar{Y}$. Then $P\bar{Y}/PX$ is defined as the cosine of the (in general not symmetric) angle of the two directions. Let $X(t_0)$ be a given point of the curve $X(t)$. In a neighborhood of $X(t_0)$ take a field of extremals containing the extremal tangent to $X(t)$ at $X(t_0)$. The curvature of $X(t)$ at $X(t_0)$ is the derivative at t_0 with respect to the arc length on $X(t)$ of the angle which the extremal of the field through $X(t)$ forms with $X(t)$.

H. Busemann (Chicago, Ill.).

Fubini, Guido. The distance in general Fuchsian geometries. Proc. Nat. Acad. Sci. U.S.A. **26**, 700–708 (1940). [MF 3370]

The general Fuchsian geometry is defined by the space R of the symmetric complex matrices $Z=X+iY$ of order n, where Y is positive-definite and the square of the line element is given by the trace of the matrix $Y^{-1}(dZ)\,Y^{-1}(d\bar{Z})$. It is proved that the geodesic distance of two points Z_1 and Z_2 of R is $(\sum_{k=1}^{n} (\log q_k)^2)^{\frac{1}{2}}$, where $q_1, \cdots, q_n$ are positive and the roots of a certain algebraic equation connected with Z_1 and Z_2. *C. L. Siegel* (Princeton, N. J.).